Autocorrelation as a source of truncated Lévy flights in foreign exchange rates

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Abstract

We suggest that the ultraslow speed of convergence associated with truncated Lévy flights [1] may well be explained by autocorrelations in data. We show how a particular type of autocorrelation generates power laws consistent with a truncated Lévy flight. Stock exchanges have been suggested to be modeled by a truncated Lévy flight [2-4]. Here foreign exchange rate data are taken instead. Scaling power laws in the "probability of return to the origin" are shown to emerge for most currencies. A novel approach to measure how distant a process is from a Gaussian regime is presented.

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1. Introduction

Financial asset prices are likely to follow non-Gaussian random walks [5]. Such random walks generally show short range autocorrelation, and the autocorrelation usually has exponential decay with very short characteristic times, a reason why most studies often treat data as pairwise uncorrelated [6]. Also, the standard deviations of price changes (volatility) are time dependent.

One benchmark study of financial markets is that of Mantegna and Stanley [2] for the Standard & Poor 500 index. A truncated Lévy flight (TLF) [1] is suggested to model the data. Such a result can also be extended to encompass emerging market indexes [3,4]. Here we further extend these studies to consider other financial assets. We employ the same method as that of the previous studies, but we reach the same results by employing a different approach. We take data on daily foreign exchange rates for 30 countries against the US dollar. Doing so, we are able to replicate some previous findings regarding the stock exchange indexes. For instance, the "probability of return to the origin" of Mantegna and Stanley is analyzed and most distributions appear to have power laws that are consistent with the presence of a Lévy distribution for their modal region; they are thus likely to be modeled by a TLF.

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A major theoretical contribution of this paper is, however, to put forward "physical" reasons to explain why a TLF emerges in that sort of data. The autocorrelation of these processes is analyzed under the assumption that such time series cannot be considered as independent and identically distributed (IID) processes. The presence of the autocorrelations is shown to be responsible for the scaling leading to the appearance of the TLF.

TLFs are characterized by an ultraslow convergence to equilibrium [1]. We argue that, in spite of possessing a characteristic time, TLFs cannot be linked to either "short range" or "long range" autocorrelation. We then develop a novel method to measure how two or more processes currently are distant from equilibrium; we also discuss which an expected time of "termalization" is. Our approach seems to be universal in that it can be extended to encompass all sorts of autocorrelated processes, and not only those described by a TLF.

We also suggest that a dynamic kurtosis can be usefully defined to measure the speed of convergence of a stochastic process to the Gaussian regime; this case is then illustrated with the currency data.

The structure of the paper is as follows. Section 2 briefly describes the data sets of foreign exchange rates to be used throughout. Section 3 analyzes the data and shows that a TLF may be consistent with the behavior of most currencies. Section 4 shows that a necessary condition for the TLF to emerge is the presence of autocorrelations in data. Section 5 shows how a dynamic kurtosis curve can be used to measure the speed of convergence of a stochastic process to the Gaussian regime. Section 6 illustrates the method presented in Section 5 with the exchange rate data. Section 7 concludes.

2. Data

The data sets employed were taken from the Federal Reserve website at http://www.federalreserve.gov/releases/H10/hist/. They refer to a currency value in US dollar terms. These foreign exchange rates were collected by the Federal Reserve Bank of New York from a sample of market participants. They are noon buying rates in New York from cable transfers payable in foreign currencies. As standard, here we ignore "holes" from weekends and holidays; analysis thus concentrates on trading days. We take the historic values of 30 currencies. Table 1 shows the currencies of the countries, historical time period, and number of datapoints.

As usual, we take returns Z rather than raw data, i.e.

$$Z_{\Delta t}(t) = Y(t+\Delta t) - Y(t)\,, \tag{1}$$

where $Y(t)$ is a rate at day t.

3. Truncated Lévy flights

This section discusses the finding of power laws in so-called "probabilities of return to the origin" $P(Z = 0)$, a fact which is consistent with the TLF [2,6]. The usual approach to characterize functional forms of a probability density function (pdf) is to evaluate their tails. The method of taking the probability of return to the origin is put forward by

Mantegna and Stanley [2]. Such an alternative method is more useful for experiments (like theirs) of taking an increasing number of values for Δt. Since larger values for Δt also mean a reduced number of datapoints, the method of investigating tails renders the determination of the parameters characterizing a pdf difficult. Taking the probability of return to the origin, by contrast, allows one to study a point of every probability distribution that is least affected by the noise coming from the finiteness of the experimental set of data. It is unclear, however, that means are fixed at zero for all Δt. Actually, the means are likely to grow and to follow a power law [4]. Since the peak of a distribution is not exactly located at $Z = 0$ for all Δt, the probability of return to the origin should be better represented by $P(W = 0) = P(Z - \omega(\Delta t)^{\beta} = 0)$, where ω and β are parameters. To experimentally find $P(0)$, a small threshold value f should be defined such that $P(0) \approx P(-f \leq W \leq f)$ [4].

To define a TLF we first consider a standard symmetric Lévy distribution, i.e.

$$P_L(x, \Delta t) = \frac{1}{\pi}\int_0^{\infty} \exp\left(-\gamma \Delta t q^{\alpha}\right)\cos(qx)dq \tag{2}$$

Secondly, we define

$$P(x) = \begin{cases} 0, x < -l \\ cP_L(x), -l \leq x \leq l \\ 0, x > l \end{cases} \tag{3}$$

for some threshold l and normalized constant c. Distribution (2) reaches equilibrium for the aggregative variables $S_{\Delta t} = \sum_{i=1}^{\Delta t} x_i$ in accordance with the central limit theorem. However, when $\Delta t \to 1$ the process can be described by a Lévy stable pdf. In short,

$$P(x) \approx \begin{cases} Lévy, \Delta t << n_x \\ Gauss, \Delta t >> n_x \end{cases} \tag{4}$$

where $n_x \approx Al^{\alpha}$ is a crossover time, and A is a constant that depends on the value of γ.

A TLF distribution is not stable and converges sluggishly to the Gaussian regime [1] (a process which is dubbed "termalization"). A presentation of other features of the TLF can be found elsewhere [6].

The probability of return to the origin in a Lévy process is given by [2,6]

$$P_{\Delta t}(0) = \frac{\Gamma(1/\alpha)}{\pi\alpha(\gamma\Delta t)^{1/\alpha}} \tag{5}$$

from which it follows that

$$\log P_{\Delta t}(0) = \log\left(\frac{\Gamma(1/\alpha)}{\pi\alpha\gamma^{1/\alpha}}\right) + (-1/\alpha)\log \Delta t \tag{6}$$

In a plot of log $P(0)$ versus log Δt a straight line of slope $-1/\alpha$ emerges within the time window for which the Lévy regime holds. The slope then approaches –0.5, which is the value corresponding to the Gaussian asymptotic equilibrium. The probability of return to the origin in a Gaussian equilibrium has the value predicted for a normal process, i.e.

$$P_{\Delta t}(0) = \frac{1}{\sqrt{2\pi}(\Delta t)^{1/2}\sigma} \tag{7}$$

Lévy stable pdfs are self-similar [6]. To usefully compare the distributions for increasing values of Δt, scaled variables are taken, i.e.

$$S_{\Delta t} = \frac{S_1}{(\Delta t)^{1/\alpha}},$$
$$P(S_{\Delta t},1) = \frac{P(S_1,\Delta t)}{\Delta t^{1/\alpha}} \tag{8}$$

Thus the data collapse onto the $\Delta t = 1$ distribution, at least for the central region of the pdf. Indeed there are departures from the Lévy stable pdf as far as the tails are concerned. Datapoints are generally lower than the ones of a Lévy pdf, which means that second moments are finite. For this very reason, the process is considered to have two regimes – a Lévy and a Gaussian – which are separated by the crossover time n_x and governed by distinct statistical properties.

As observed, the TLF is used to model the S&P 500 index [2,6] as well as emerging stock markets [3,4]. Such papers argue that the TLF describes the asymptotic price change distributions measured at distinct time horizons as well as their scaling properties. However, as also observed, financial price changes are unlikely to be stochastic process with IID increments. For this reason, we make a case for the TLF to be present only in short range autocorrelated processes.

Here we extend such studies to consider the currencies presented in Section 2. We employ the same method as that of the previous studies. However, we reach the same results in Section 4 by employing a different approach.

Fig. 1 displays the logarithm of the pdfs of currency returns for selected countries in Table 1, namely Australia, Britain, Canada, Belgium, India, Brazil, China, and South Africa. Increases in time horizons range from $\Delta t = 1$, 2, and 5 trading days (a week) to 240 trading days (a year). A spreading of the pdfs characteristic of any random walk is observed. The pdfs are roughly symmetric for the currencies of developed countries, but there is marked skewness for the currencies of emerging markets.

Fig. 2 shows a log-log plot of $P(0)$ against Δt. Roughly, scaling power laws emerge for the currencies within the time window of $1 \leq \Delta t \leq 100$; and this is consistent with the presence of a TLF.

By plugging the slope value of $-1/\alpha$ we get γ from (2). By using (5), the data are made to collapse onto the $\Delta t = 1$ distributions of the currencies (Fig. 3).

Table 2 presents parameters α and γ for the currencies in Table1. Parameter α is greater than 2 for six countries, namely Canada, China, Malaysia, South Africa, Thailand, and Venezuela; the currencies of these countries may (or may not) be outside the Lévy stable regime. For all the other currencies, a TLF may describe the data within a time window of (generally) 100 trading days (not shown).

Thus a Lévy stable pdf with finite second moments may model the modal region of such processes within a finite time interval. The presence of the TLF is arguably pervasive in daily time series of currency returns.

Previous work on the presence of TLFs in financial series focuses analysis on the value of the crossover time n_x and its dependence on α and γ, as well as on the volatility of α. This is not our primary concern in here. Rather, we are interested to know what the "physical" reasons (if any) for the appearance of the TLF are. As seen, financial indexes cannot be modeled in terms of stochastic processes with IID increments, even if a process is short range autocorrelated [1]. How then is it possible for a stochastic process to fit, at least for a finite time, data which are not independent?

Another concern of this paper is to investigate the property of the TLF of ultraslow convergence to the Gaussian regime. Here we look for the reasons of such a “friction”. Does sluggish convergence emerge for distributions other than the TLF? If the answer is yes, then how can we measure the expected time of termalization? Next section deals with these questions.

4. The origins of the TLF

The TLF is a stochastic process defined for independent variables $\{x\}$. However, real world data often present autocorrelation. But for short range autocorrelated data (e.g. financial data), it is usual to treat them as independent after their “characteristic” time has elapsed. The purpose of this section is to show how an autocorrelation function, even if it is at the noise level, can lead to scaling laws that are compatible with the presence of the TLF. We also demonstrate how the correlation acts as a friction causing the ultraslow convergence to the Gaussian regime**.**

For Δt = 1, 2, 3,..., consider a sum variable $S_{\Delta t}$, i.e.

$$S_{\Delta t} = x_1 + \cdots + x_{\Delta t} \tag{9}$$

and a condition of identically (but not necessarily independent) distributed variables, i.e.

$$f(x_1) = f(x_2) = \cdots = f(x_{\Delta t}) \tag{10}$$

Notice that the aggregative variable $S_{\Delta t}$ is equivalent to $Z_{\Delta t}(t) = Y(t+\Delta t) - Y(t)$. We now define a “central variable” as

$$S'_{\Delta t} = S_{\Delta t} - < S_{\Delta t} > \tag{11}$$

and a “reduced variable” as

$$\overline{S}_{\Delta t} = S'_{\Delta t} / \sigma_{\Delta t}; \ \ \sigma_{\Delta t} = \sqrt{< S^2_{\Delta t} > - < S_{\Delta t} >^2} \tag{12}$$

Lévy [9] originally proved that the characteristic function $\varphi(q)$ of a process with finite variance obeys the following:

$$\varphi(0) = 1, \varphi'(0) = 0, \varphi''(0) = 1,$$

$$\psi = \log \varphi(q) = -\frac{q^2}{2} + a_3 q^3 + a_4 q^4 + ... = -\frac{q^2}{2}(1 + w(q)) \Rightarrow \tag{13}$$

$$\varphi(q) = e^{-\frac{q^2}{2}(1+w(q))}, w(0) = 0$$

The characteristic function of our reduced variable (12) can then be written as

$$\overline{\varphi}_{\Delta t}(q) = e^{-\frac{q^2}{2}(1+w_{\Delta t}(q))} \tag{14}$$

In turn, the characteristic function of our central variable (11) can be written as

$$\varphi'(q) = \overline{\varphi}_{\Delta t}(\sigma_{\Delta t} q) = e^{-\frac{(\sigma_{\Delta t} q)^2}{2}(1+w_{\Delta t}(\sigma_{\Delta t} q))} \tag{15}$$

We state the following.

Definition 1. A *stable process* occurs in the interval $[\Delta t_1, \Delta t_2]$ if

$$w_{\Delta t}(q) = w(q), \quad \forall \Delta t_1 \le \Delta t \le \Delta t_2 \tag{16}$$

It is worth noticing that Definition 1 does not guarantee stability for the distributions $f(x_i)$. Actually the most interesting results are those with processes whose distributions are not stable. However, Definition 1 does imply that the characteristic function is constant within the interval [Δt_1, Δt_2], i.e.

$$\overline{\varphi}_{\Delta t}(q) = \overline{\varphi}(q), \quad \Delta t_1 \le \Delta t \le \Delta t_2 \tag{17}$$

Put it another way, the reduced variables $\overline{S}_{\Delta t}$ have the same pdfs for Δt within the interval [Δt_1, Δt_2]; and this is equivalent to the scaling of statistical laws. There is a difference, though. Here the distribution laws $f(x_i)$ are not necessarily stable, i.e. they are not Lévy distributions with $\alpha \le 2$. For a process to be stable in the sense above and, at the same time, for the probabilistic laws $f(x)$ to be unstable, autocorrelations between variables x_i must be present. Indeed, if a process is independent, the central limit theorem implies that

$$\begin{aligned} &\psi_{S_{\Delta t}}(q) \to \Delta t\, \psi\left(\frac{q}{\sqrt{\Delta t}}\right) = -\frac{q^2}{2} + a_3 \frac{q^3}{\sqrt{\Delta t}} + a_4 \frac{q^4}{\Delta t} + \ldots \\ &\varphi_{S_{\Delta t}}(q) \to \varphi_{S_{\Delta t}}(q/\sqrt{\Delta t}) = e^{-q^2/2\left(1+w\left(\frac{q}{\sqrt{\Delta t}}\right)\right)} \end{aligned} \tag{18}$$

where $\psi = \log \varphi$. The central limit theorem follows [9] because

$$\Delta t \to \infty \Rightarrow \varphi_{S_{\Delta t}}(q/\sqrt{\Delta t}) \to e^{-q^2/2(1+w(0))} = e^{-q^2/2} \tag{19}$$

and $w(0) = 0$. From equation (18) we can see that (16) is only satisfied if there is a friction preventing the convergence $w_{\Delta t}(q) \to 0$ as $\Delta t \to \infty$. Such a friction is precisely the one found in autocorrelated processes.

An interesting property of a stable process is scaling in the probability $P(S_{\Delta t} =< S_{\Delta t} >)$ with standard deviation $\sigma_{\Delta t}$. Consider

$$P(S_{\Delta t} =< S_{\Delta t} >) = P(S'_{\Delta t} = 0) \tag{20}$$

From the definition of a characteristic function it follows that

$$P(S'_{\Delta t} = 0) = \int_{-\infty}^{\infty} \varphi'(q)\, dq \tag{21}$$

Notice that equation (15) implies that

$$P(S'_{\Delta t} = 0) = \int_{-\infty}^{\infty} \overline{\varphi}_{\Delta t}(\sigma_{\Delta t} q)\, dq, \Delta t_1 \le \Delta t \le \Delta t_2 \tag{22}$$

By performing the transformation $q \to \sigma_{\Delta t} q$ we obtain

$$P(S'_{\Delta t} = 0) = \frac{1}{\sigma_{\Delta t}} \int_{-\infty}^{\infty} \overline{\varphi}_{\Delta t}(q')\, dq', \Delta t_1 \le \Delta t \le \Delta t_2 \tag{23}$$

By taking (20) into account, we have

$$P(S'_{\Delta t} = 0) = \frac{1}{\sigma_{\Delta t}} \int_{-\infty}^{\infty} \overline{\varphi}(q')\, dq' \tag{24}$$

Since the process is stable, the integral above equals a constant A; thus

$$\begin{aligned} &P(S'_{\Delta t} = 0) = \frac{1}{\sigma_{\Delta t}} \int_{-\infty}^{\infty} \varphi(q')\, dq' \Rightarrow \\ &P(S'_{\Delta t} = 0) = P(S_{\Delta t} =< S_{\Delta t} >) = \frac{A}{\sigma_{\Delta t}} \end{aligned} \tag{25}$$

To summarize: the probability of return to the origin is governed by a power law within the interval of stability [Δt_1, Δt_2], which is the inverse of the standard deviation $\sigma_{\Delta t}$. Moreover, if the standard deviation scales as a power law of the type

$$\sigma_{\Delta t} = C\Delta t^{1/\alpha}, \qquad \Delta t_1 \le \Delta t \le \Delta t_2 \tag{26}$$

then we have

$$\begin{aligned} P(0) &= \left(\frac{A}{C}\right)\frac{1}{\Delta t^{1/\alpha}} \Rightarrow \\ \log P(0) &= \log\left(\frac{A}{C}\right) + \left(\frac{-1}{\alpha}\right)\log \Delta t \end{aligned} \tag{27}$$

Notice that (27) is similar to equation (6) as long as we consider the identification

$$\begin{cases} \left(\dfrac{\Gamma(1/\alpha)}{\pi\alpha\gamma^{1/\alpha}}\right) = \dfrac{A}{C} \\ n = \Delta t \end{cases} \tag{28}$$

Let us now suppose that $w_{\Delta t}(q)$ varies very slowly within the interval [Δt_1, Δt_2]. Such a process can be dubbed "quasi-stable". Here a relation similar to (25) holds i.e.

$$P(0) = \left(\frac{A_{\Delta t}}{\sigma_{\Delta t}}\right) \tag{29}$$

where $A_{\Delta t}$ depends on the value of Δt:

$$A_{\Delta t} = \int_{-\infty}^{\infty} \overline{\varphi}_{\Delta t}(q')dq' = \int_{-\infty}^{\infty} e^{-q^2/2(1+w_{\Delta t}(q))}dq \tag{30}$$

Since $w_{\Delta t}$ is almost constant in the interval [Δt_1, Δt_2], $A_{\Delta t}$ varies very slowly. If the standard deviation $\sigma_{\Delta t}$ obeys the scaling law (26), we have

$$\begin{aligned} P(0) &= \left(\frac{A_{\Delta t}}{C}\right)\frac{1}{\Delta t^{1/\alpha}} \Rightarrow \\ \log P(0) &= \log\frac{A_{\Delta t}}{C} - \frac{1}{\alpha}\log \Delta t \end{aligned} \tag{31}$$

Since $A_{\Delta t}$ is almost constant, we can write $A_{\Delta t} \approx$ A and obtain the functional relation

$$P(0) \cong \left(\frac{A}{C}\right)\frac{1}{\Delta t^{1/\alpha}} \tag{32}$$

If $\alpha \le 2$ then the probability of return to the origin follows a scaling which is similar to a stable distribution and is characterized by the characteristic function

$$\varphi(q) = e^{-k|q|^{\alpha}} \tag{33}$$

Therefore, the central region of the distribution follows a Lévy stable process, even if the distributions $f(x)$ themselves are not either stable, independent, or generated by a Lévy stochastic process.

It is worth emphasizing the following.

(1) That the standard deviation obeys (26) is not necessary. Other relations beyond the power law might exist, in which cases a TLF will not appear.

(2) Even if (26) holds, it is not necessary that $\alpha \le 2$. Indeed if $\alpha > 2$ the central region of the distribution is not fitted by a Lévy stable pdf.

(3) The presence of correlations is responsible for both the quasi-stability of a process and the scaling compatible with a TLF. By no means the process itself is generated by a Lévy stochastic process, since it is not only autocorrelated but also the pdfs $f(x_i)$ are not stable.

(4) In a quasi-stable process there exists a time horizon in which $w_{\Delta t}(q)$ reaches zero very slowly because the autocorrelation acts as a friction. This is the reason why the ultraslow convergence associated with the TLF emerges. However, as $w_{\Delta t}(q) \cong 0$ the characteristic function is closer to the Gaussian function

$$\overline{\varphi}(q) \approx e^{-q^2/2} \quad (w_{\Delta t} \approx 0, \Delta t \geq n_x)$$

in which case $P(0)$ rescales as predicted for a normal distribution, i.e.

$$P(0) \cong \frac{A}{\Delta t^{1/2}}$$

Thus, the fitting (31) does hold beyond some threshold n_x where $\alpha = 2$.

(5) The convergence time of the process is given by (31).

(6) None of the results above holds true if $w_{\Delta t}$ is not sluggish. It is thus implied that there exists particular types of autocorrelation associated with TLFs. It is still unsettled, though, which kind of autocorrelation is compatible with a TLF.

(7) For some types of autocorrelation the standard deviation of a process is not governed by power laws. In such cases, a TLF does not emerge. The TLF does not appear as well if the condition of quasi-stability is not fulfilled.

5. Kurtosis and convergence

We now show how a dynamic kurtosis curve can be usefully defined to evaluate the convergence toward the Gaussian regime regardless of the TLF. Focusing analysis only on the kurtosis of a process has an advantage of not being necessary to have power laws in either $P(0)$ or the variance. Moreover, the method can be used to compare two or more processes.

Thus let us first define $S_{\Delta t}^R$ as the sum of Δt variables taken randomly from a time series for $S_1 = Z_1(t)$ and, secondly, let us define $S_{\Delta t}^o$ as the sum of Δt variables as usual, i.e. $S_{\Delta t}^o(1)$ is the sum of the first Δt values of $Z_1(t)$, $S_{\Delta t}^o(2)$ is the sum of next Δt, and so on.

Variables $S_{\Delta t}^R$ are defined such that we expect them to be pairwise independent. We also expect $S_{\Delta t}^o$ to be short range autocorrelated.

Function $w(q)$ as defined in (13) can be written as [9]

$$w(q) = w_R + iw_i,$$

$$\begin{cases} w_R = -\dfrac{1}{12}(\mu^4 - 3)q^2 - (\dfrac{1}{24}q^4 + \dfrac{q^3}{36} - \dfrac{1}{12})q^4 + \ldots \\ w_R = -\dfrac{1}{12}(K)q^2 + O(q^4) \end{cases} \tag{34}$$

where $\mu^{\Delta t} = \sum \left[\dfrac{x - < x >}{\sigma} \right]^{\Delta t}$. Here we take the definition of kurtosis as usual, i.e.

$$K = \frac{1}{N}\sum_{i=1}^{N}[(x_i - \langle x \rangle)/\sigma]^4 - 3 \tag{35}$$

For the imaginary part of $w(q)$ we similarly have

$$\begin{cases} w_i = \frac{1}{12}(\mu^3 - \mu^5/10)q^5 + \frac{1}{6}\mu^3 q^3 + O(q^7) \\ w_i = \frac{1}{6}(J)q^3 + O(q^5) \end{cases} \tag{36}$$

where we use the standard definition of skewness as

$$J = \frac{1}{N}\sum_{i=1}^{N}\left[\frac{x_i - \langle x \rangle}{\sigma}\right]^3 \tag{37}$$

Since $w(q)$ can be seen as a measure of how distant the characteristic function is from the equilibrium given by $\varphi(q) = e^{-q^2/2}$ (Gaussian regime), the leading term in $w(q)$ (i.e. the kurtosis) gives a good measure of the distance from equilibrium. For aggregative variables, function $w(q)$ reaches zero sluggishly. The convergence speed is $1/\sqrt{\Delta t}$ for independent processes; for autocorrelated processes it is much slower. Then the dynamic behavior of $w(S_{\Delta t}) = w(Z_{\Delta t}(t))$ is a good candidate for a measure of the convergence speed, *whatever the autocorrelation of a system*.

6. An illustration with currency data

To illustrate our approach we take the British pound and the Indian rupee. Fig. 4 and Fig. 5 show the characteristic functions of $S_{\Delta t}^{R}$ and $S_{\Delta t}^{o}$ as a function of Δt. As expected for an independent process, $S_{\Delta t}^{R}$ reaches the Gaussian regime (where $\varphi(q) = e^{-q^2/2}$) for $\Delta t \approx 50$. Fig. 5 displays the slow convergence of $S_{\Delta t}^{o}$ to equilibrium. Fig. 6 is a plot of the scaled second moment versus Δt for both variables together with the curve of an IID process ($\sigma_{S_{\Delta t}} = \sigma\sqrt{\Delta t}$). Process $S_{\Delta t}^{R}$ exactly fits the curve of the IID process, whereas process $S_{\Delta t}^{o}$ does not. This means that there is an autocorrelation in the process which is responsible for such a behavior. Indeed:

$$\begin{aligned} {\sigma_{S_{\Delta t}}}^2 &= \Delta t \sigma^2 + 2\sigma^2\left(\sum_{i,j=1}^{\Delta t}(\langle x_i x_j \rangle - \langle x_i \rangle\langle x_j \rangle)/\sigma^2\right) \\ \sigma_{S_{\Delta t}} &= \sigma\sqrt{\Delta t + 2\sum_{\substack{i,j \\ i\neq j}}^{\Delta t} corr(x_i, x_j)} = \sigma\sqrt{\Delta t + \sum_{i,j}^{\Delta t} corr(x_i, x_j)} \end{aligned} \tag{38}$$

Fig. 7 presents the autocorrelation functions for the pound and rupee. Regarding the pound, the autocorrelation starts out at 0.06 and continues to be very low throughout; however, it neither dies away nor experiences the exponential decay of short range autocorrelated processes. There is no "characteristic time". Nevertheless, the label "long range autocorrelated" is not appropriate either, since the autocorrelation is low from the start. The autocorrelation seems to be always at the noise level, although it is still responsible for both the particular curve followed by $S^o_{\Delta t}$ and the scaling laws leading to the TLF.

The effect of the autocorrelation can be appreciated by an interesting property relating both the $S^o_{\Delta t}$ and $S^R_{\Delta t}$ processes as follows. We first take points on the curve of $S^R_{\Delta t}$ (which follows the power law of an independent process). We then add up $\sum corr(x_i, x_j)$ to every point. The "adjusted" autocorrelation curve $\sigma^R_{S_{\Delta t}}$ precisely matches the curve of $\sigma^o_{S_{\Delta t}}$ (Fig. 8).

It is worthwhile noticing that the autocorrelation function for the pound can assume the familiar shape of a short range autocorrelated process for high frequency data. Most financial variables are characterized by very short time memory (a few trading minutes in most cases); the correlations are thus expected to fall off to the noise level after just a trading day. Autocorrelation at the noise level has usually made a number of researchers to treat data as pairwise independent. However, we argue that such processes cannot be treated as stochastic as the "noise" correlation is itself responsible for many of the system properties, such as the typical power law scaling of the TLF as well as its sluggish speed of convergence.

Thus a process $S^o_{\Delta t}$ cannot be treated as independent even though the autocorrelation function is always at the noise level. Second moments are governed by scaling laws, a fact which is responsible for the particular power law in $P(0)$ of the TLF.

Fig. 9 shows the typical behavior of $w(q)$ for both the pound and rupee. As far as $S^o_{\Delta t}$ is concerned, there is a clear "resistance" preventing the process to reach equilibrium at $w = 0$. By contrast, variable $S^R_{\Delta t}$ reaches $w = 0$ much faster. The friction occurring in $S^o_{\Delta t}$ is consistent with both a process with memory and the scaling in the probability of return to the origin. The physical origins of such a friction are the low autocorrelation of a process which is always at the noise level. Fig. 10 presents the same information in terms of kurtosis. The curve $1/\Delta t$ of an IID process is displayed for comparison. Notice that $K_{\Delta t} = K_1/\Delta t$ for the IID process.

The presence of power laws in the second moment leads to the power law in the probability of return to the origin which is compatible with the TLF. For instance, for the British pound we have

$$\begin{aligned} &\sigma^o_{S_{\Delta t}} = 0.9770065445\Delta t^{0.5490373589} \\ &\frac{1}{\alpha} = 0.5490373589 \Rightarrow \alpha = 1.821369682 \end{aligned} \tag{39}$$

where Δt ranges from 1 to 50. The scaling breaks down for values of Δt greater than 50. As expected from our previous discussion, α is in good agreement with that in Table 2. We checked for the scaling in the second moment for all currencies in Table 1 and found that results match those presented in Section 3.

It is worth emphasizing that the scaling in the second moment is not enough for a TLF to be present. Indeed the Chinese yuan illustrates this. The yuan does not seem to be described by a TLF (Fig. 3 and Table 2), despite the fact that there is scaling in the second moment for process $S^o_{\Delta t}$ (Fig. 11).

Fig. 12 shows the behavior of $w(q)$ for the Chinese yuan. Both $S^{o}_{\Delta t}$ and $S^{R}_{\Delta t}$ follow a stochastic process. Therefore $w(q)$ does not exhibit the quasi-stability necessary for the TLF to emerge.

Fig. 13 shows that the pound starts out at $\Delta t = 1$ much closer to equilibrium ($w = 0$) than the rupee does. However, the speed of convergence is greater for the rupee (Fig. 14); this possibly happens because function w decays faster for the rupee, as measured by the dynamic kurtosis curve. Yet the process for the rupee is always more distant from equilibrium than that for the pound.

The standard interpretation sees kurtosis as a measure of the peak of a distribution. Here kurtosis receives the status of a dynamic variable measuring the distance from equilibrium every time. Kurtosis is a measure of the speed of convergence.

This section can be summarized as follows. A comparison between $S^{o}_{\Delta t}$ and $S^{R}_{\Delta t}$ can determine several statistical properties of a system, such as the scaling power laws compatible with the TLF as well as the convergence speed toward the Gaussian regime. Autocorrelation functions seem to play an important role in the scaling laws governing foreign exchange rates. The correlation of pairs that seems to be at the noise level cannot be discarded and, accordingly, a process cannot be treated as independent for that very reason. There seems to exist physical reasons for the TLF to emerge; it is the result of a particular type of low autocorrelation which is present in a process.

7. Conclusion

This paper argues that the ultraslow speed of convergence associated with truncated Lévy flights may be explained by autocorrelations in data. Daily foreign exchange rate data for 30 currencies against the US dollar are taken to illustrate that. Scaling power laws in the probability of return to the origin seem to be pervasive in currencies, a fact which is consistent with a Lévy stable stochastic process describing the modal region of their distributions.

A major theoretical contribution of the paper is to suggest physical reasons to explain why a TLF generally emerges in financial series. The autocorrelations of the exchange rate series are analyzed to show that such sort of time series cannot be considered as independent and identically distributed processes. Also, the presence of the autocorrelations is shown to be responsible for the scaling leading to the emergence of the TLF.

We argue too that, in spite of possessing a characteristic time, the TLF cannot be linked to either short range or long range autocorrelation. We then present a novel method to compare two or more processes by taking into account how distant they currently are from the Gaussian regime; we also discuss which an expected time of termalization to the Gaussian equilibrium is. Our method is supposed to be universal in that it can be extended to encompass all types of autocorrelated processes, and not only those described by the TLF.

By using the data for currencies, we also suggest to dynamically reinterpret kurtosis as a measure for the speed of convergence of a stochastic process toward the Gaussian regime.

Acknowledgements

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Country	Currency	Time Period	Datapoints
Australia	Australian Dollar	4Jan71 – 30Jun00	7392
Austria	Shilling	4Jan71 – 31Dec98	6999
Belgium	Belgian Franc	4Jan71 – 31Dec98	7013
Brazil	Real	2Jan95 – 31Dec01	1636
Britain	Pound	4Jan71 – 31Aug01	7695
Canada	Canadian Dollar	4Jan71 – 31Aug00	7445
China	Yuan	2Jan81 – 29Dec00	4963
Denmark	Krone	4Jan71 – 31Aug00	7442
Finland	Markka	4Jan71 – 31Dec98	6976
France	Franc	4Jan71 – 31Dec98	7021
Germany	Deutsche Mark	4Jan71 – 31Dec98	7021
Ireland	Pound	4Jan71 – 31Dec98	7021
India	Rupee	2Jan73 – 29Dec00	7018
Italy	Lira	4Jan71 – 31Dec98	7020
Japan	Yen	4Jan71 – 31Oct01	7729
Malaysia	Ringgit	4Jan71 – 31Oct01	7713
Mexico	Peso	8Nov93 – 31Oct01	2003
Netherlands	Guilder	4Jan71 – 31Dec98	7021
New Zealand	New Zealand Dollar	4Jan71 – 31Oct01	7719
Portugal	Escudo	2Jan73 – 31Dec98	6518
Singapore	Singapore Dollar	2Jan81 – 31Oct01	5234
South Africa	Rand	4Jan71 – 31Oct01	7708
South Korea	Won	13Apr81 – 31Oct01	5119
Spain	Peseta	2Jan73 – 31Dec98	6521
Sri Lanka	Rupee	2Jan73 – 31Oct01	6875
Sweden	Krona	4Jan71 – 31Oct01	7734
Switzerland	Swiss Franc	4Jan71 – 31Oct01	7735
Taiwan	Taiwan Dollar	30Oct83 – 31Aug01	4211
Thailand	Baht	2Jan81 – 31Jul01	5114
Venezuela	Bolivar	2Jan95 – 31Oct01	1716

Table 1. Description of Data Sets

Country	α	γ
Australia	1.41487	.004656830
Austria	1.90185	.000010368
Belgium	1.56042	.009849513
Brazil	.89059	.003707604
Britain	1.76454	.000078676
Canada	*2.04822*	.000005979
China	*4.19286*	2.5306E-11
Denmark	1.39021	.002288440
Finland	1.75114	.00040350
France	1.48668	.001021003
Germany	1.54737	.000330146
Ireland	1.61516	.000150068
India	1.87979	.000018721
Italy	1.27801	.014513000
Japan	1.43542	.011937000
Malaysia	*2.78363*	.000000169
Mexico	1.60305	.000923330
Netherlands	1.55999	.00034082
New Zealand	1.87623	.000011118
Portugal	1.33192	.00514115
Singapore	1.81272	.000027149
South Africa	*3.46313*	1.7417E-8
South Korea	.93298	.015343000
Spain	1.28282	.01962400
Sri Lanka	1.22370	.000898138
Sweden	1.53611	.000880928
Switzerland	1.68564	.000209469
Taiwan	1.19228	.003356487
Thailand	*2.03006*	.000033537
Venezuela	*4.13507*	6.6404E-9

Table 2. Parameters α and γ for the Currencies in Table1.

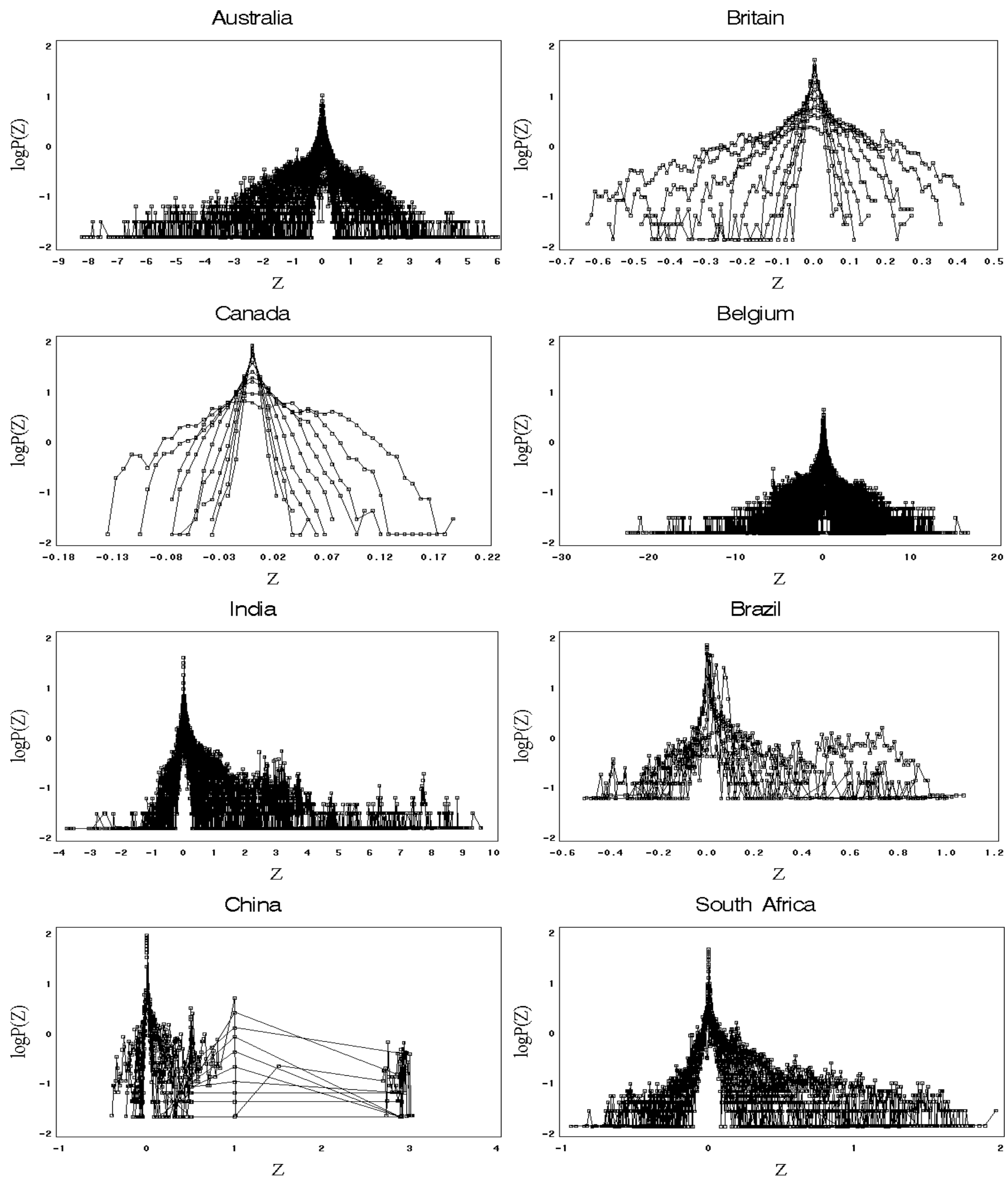


Fig. 1. Probability density functions of the currency returns of Australia, Britain, Canada, Belgium, India, Brazil, China, and South Africa observed at time intervals Δt, which range from 1 to 240 trading days. As Δt is increased, a spreading of the probability distribution characteristic of any random walk is observed.

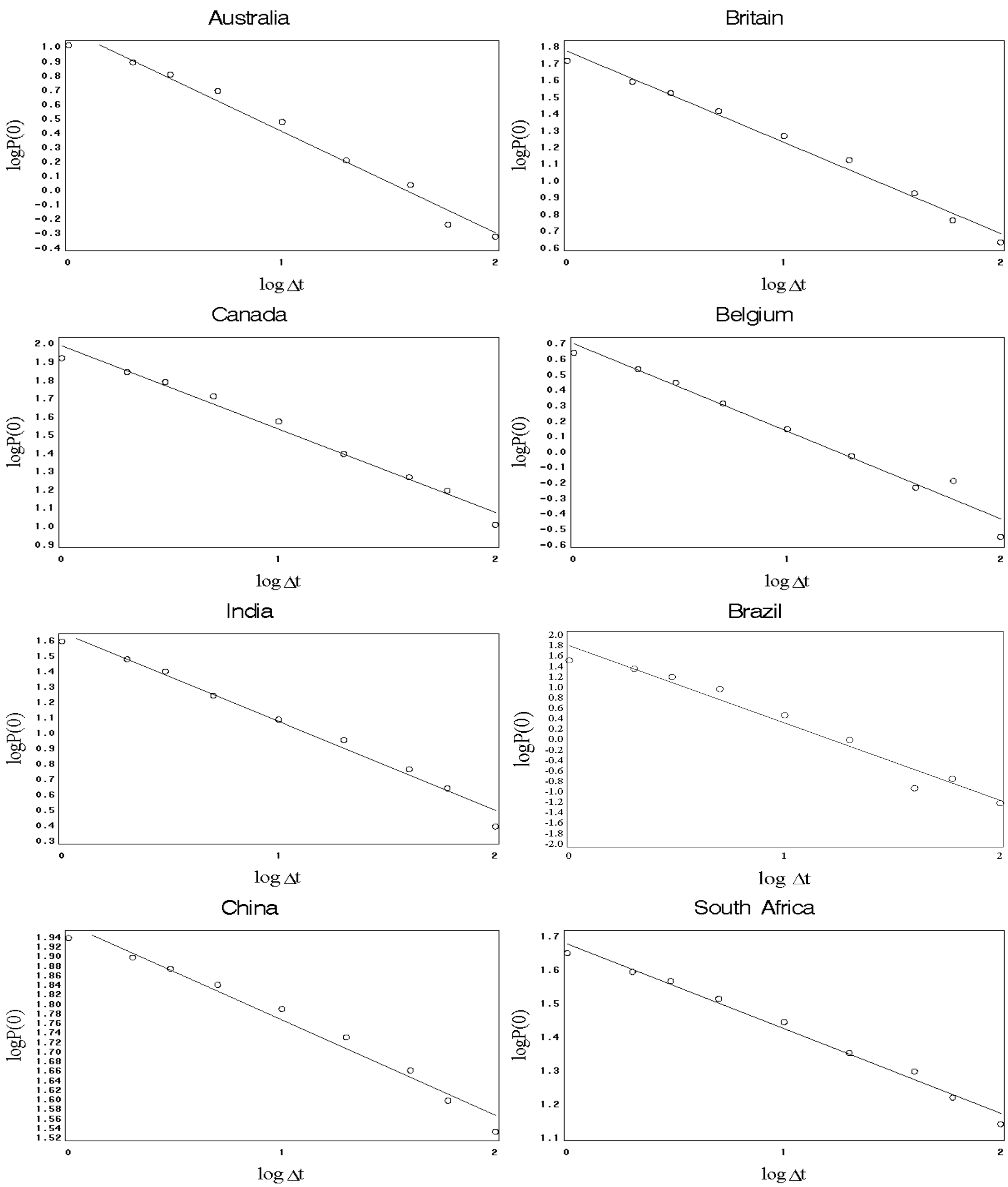


Fig 2. Log-log plot of the probability of return to the origin $P(0)$ against the time lag Δt for the currency returns of Australia, Britain, Canada, Belgium, India, Brazil, China, and South Africa. Power laws emerge within the time window of $1 \le \Delta t \le 100$. This non-Gaussian scaling is consistent with the presence of a TLF.

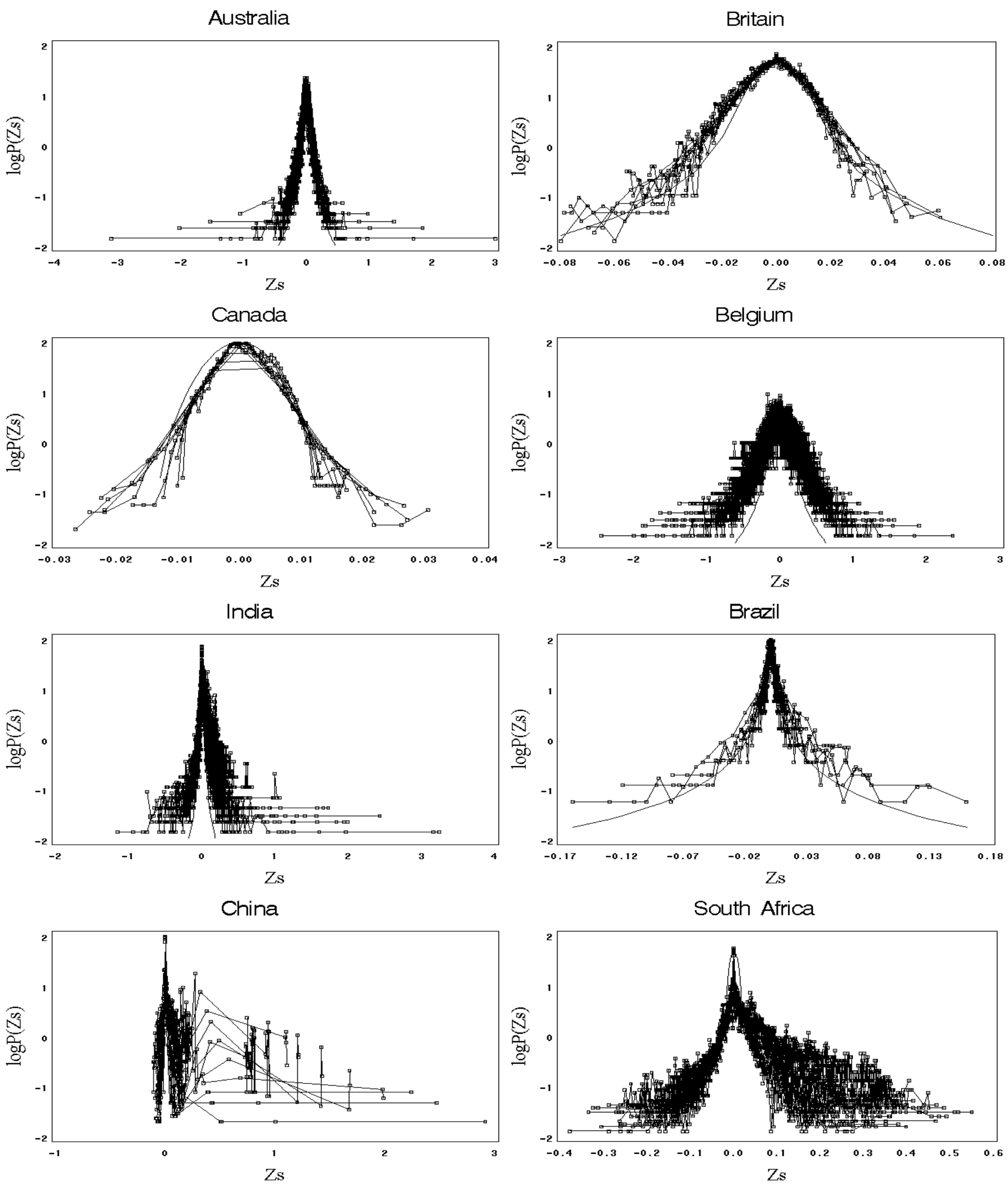


Fig. 3. The same pdfs as in Fig. 2, but now plotted in scaled units $P(Z)$. Given the scaling index α for each currency, all the data is made to collapse onto a $\Delta t = 1$ distribution.

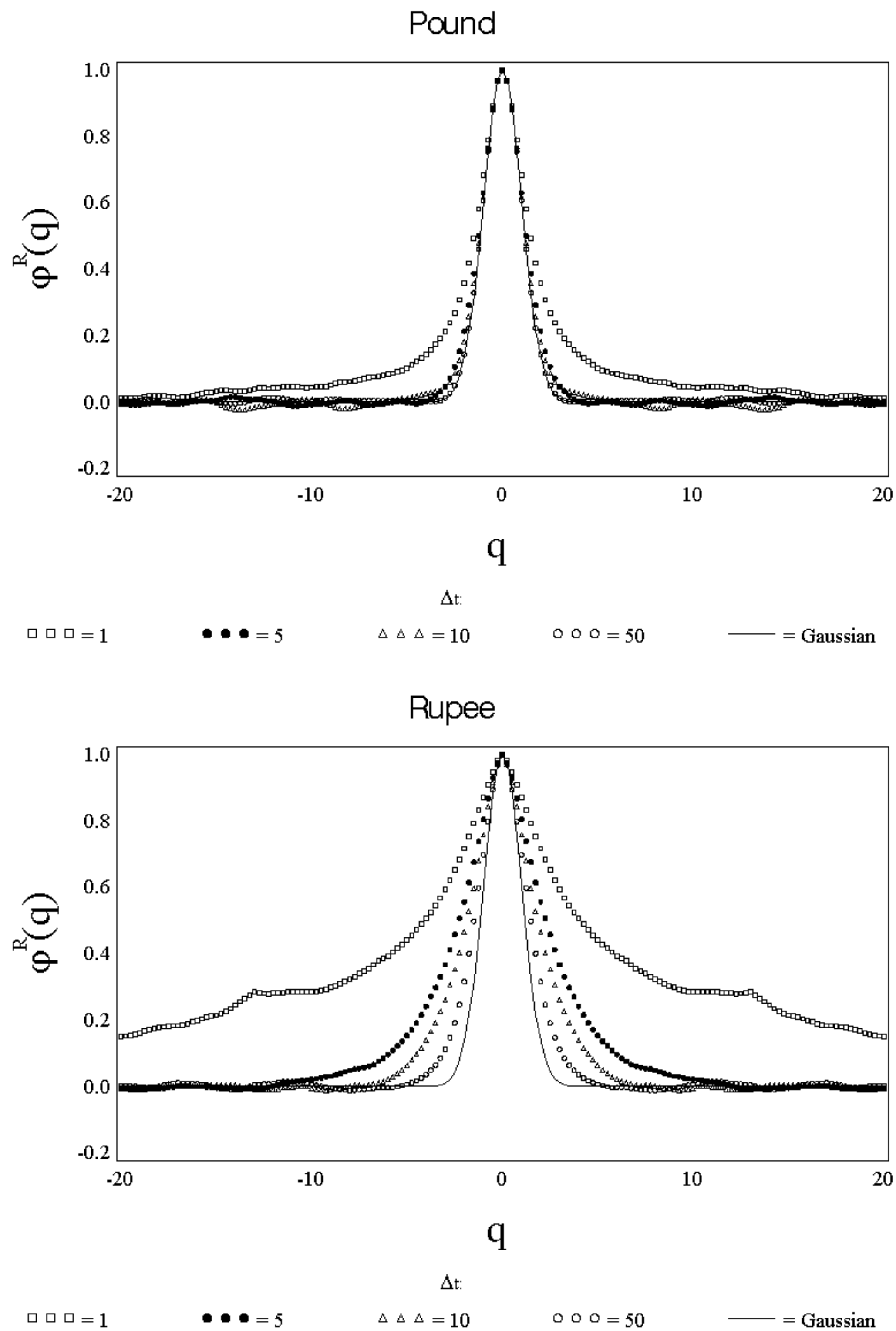


Fig. 4. Real part of the characteristic function $\varphi^R(q)$ as a function of q for the British pound and Indian rupee for variable $S_{\Delta t}^R$. The Gaussian function $\varphi^R(q) = e^{-q^2/2}$ is also displayed for the sake of comparison.

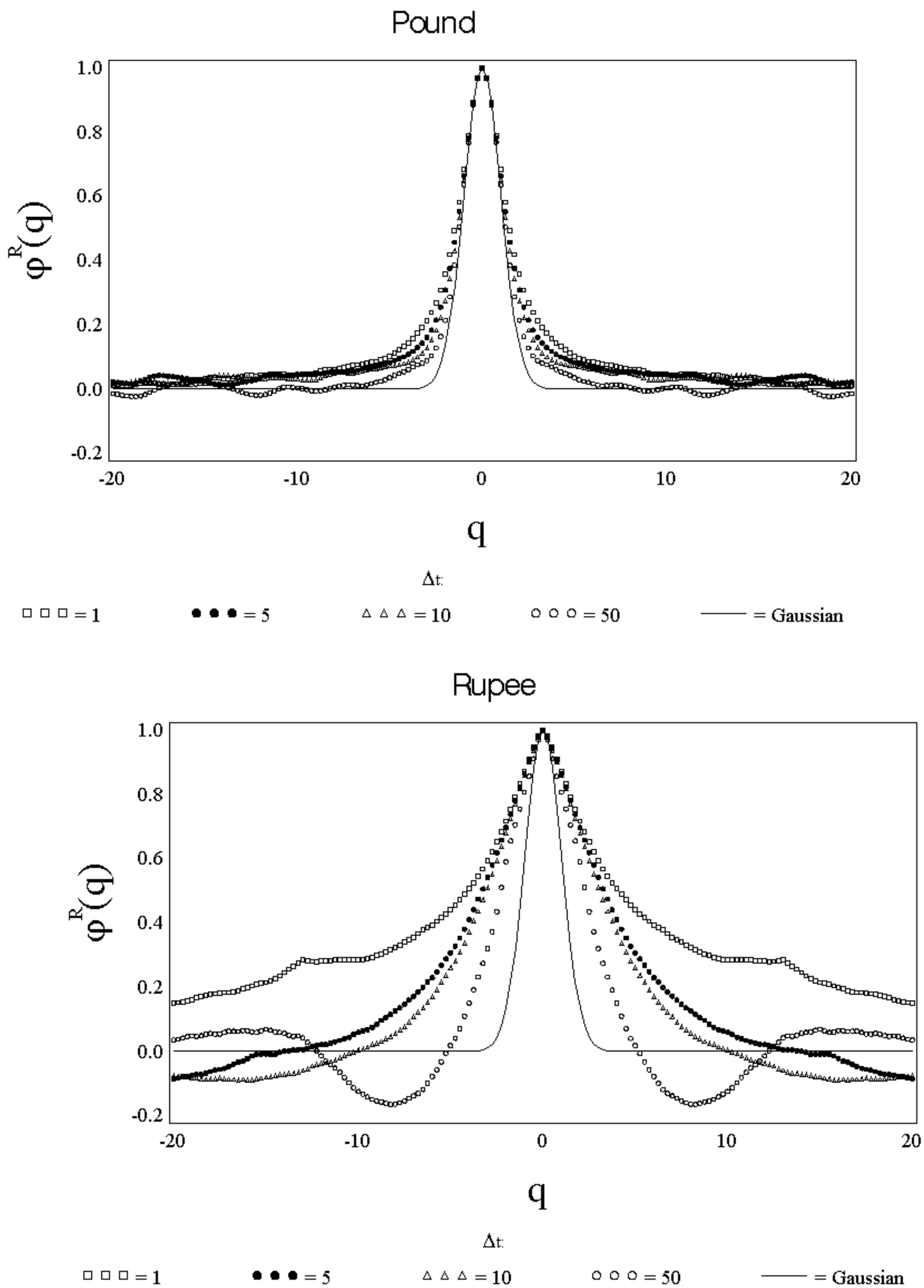


Fig. 5. Real part of the characteristic function $\varphi^R(q)$ as a function of q for the British pound and Indian rupee for variable $S^o_{\Delta t}$. The Gaussian function $\varphi^R(q) = e^{-q^2/2}$ is shown for comparison.

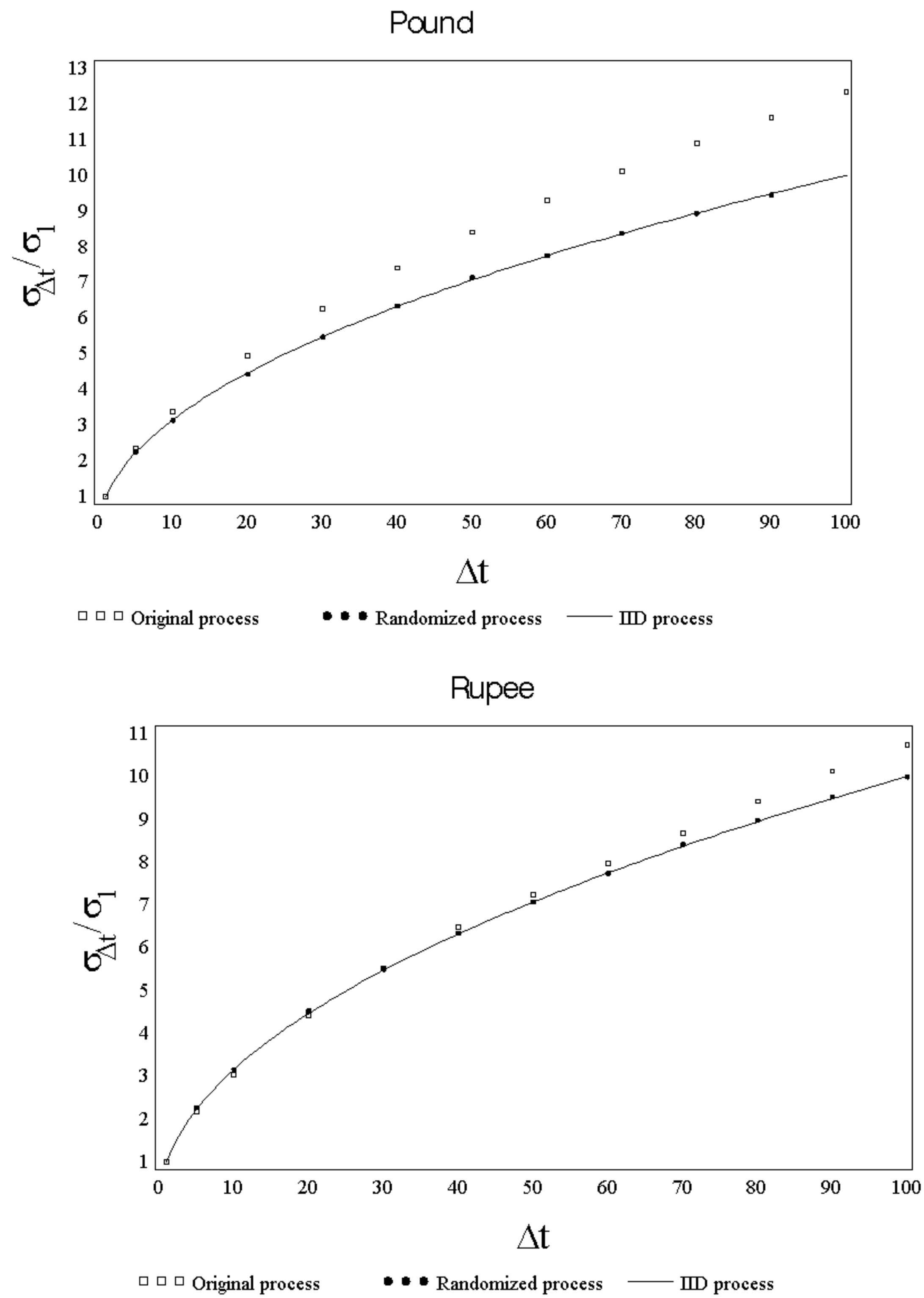
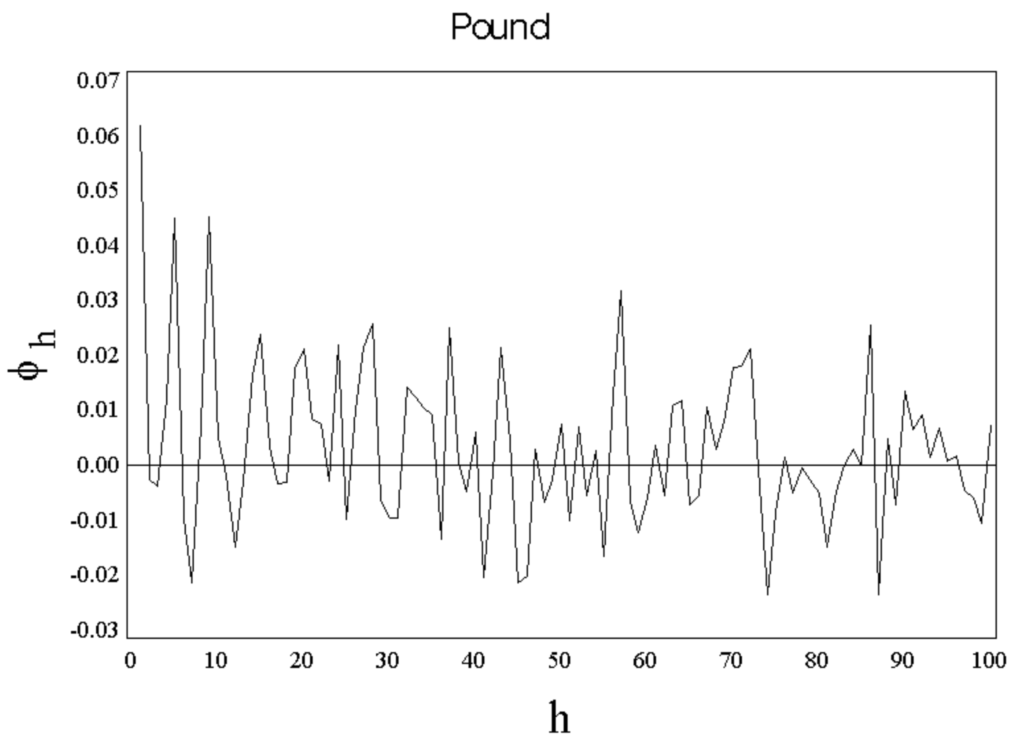


Fig. 6. Plot of the scaled second moment versus Δt for $S_{\Delta t}^{R}$ (randomized process) and $S_{\Delta t}^{o}$ (original process). The curve of the IID process $\sigma_{\Delta t} = \sigma\sqrt{\Delta t}$ is shown for comparison.



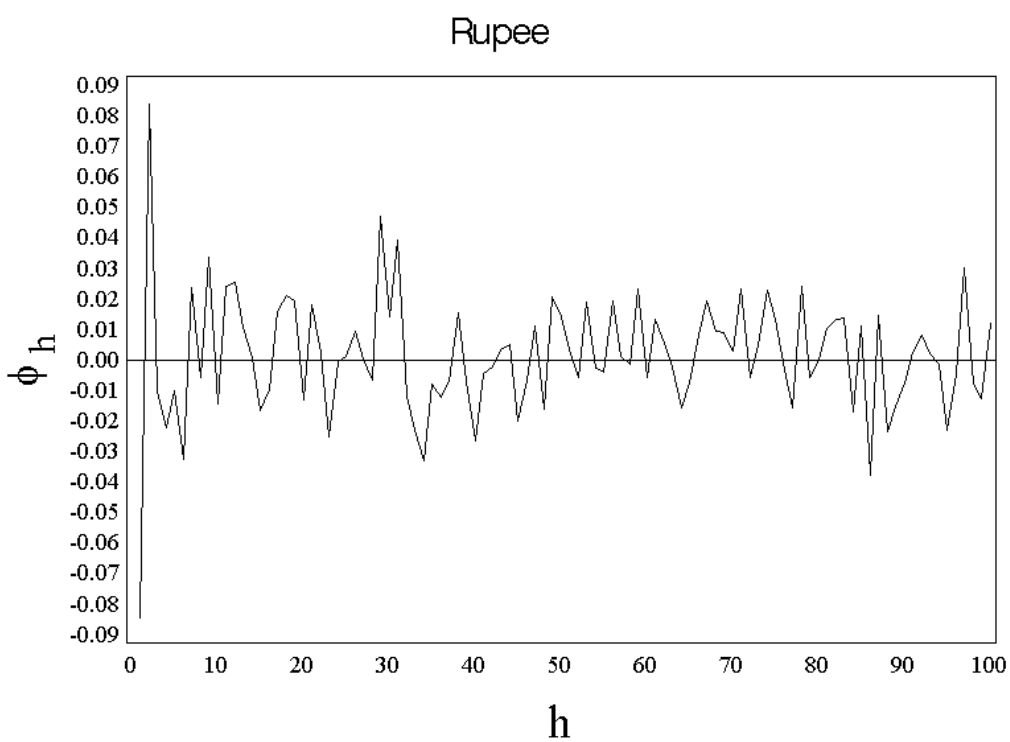


Fig. 7. Autocorrelation functions $\phi_h = \mathrm{corr}[Z_1(t), Z_1(t+h)]$ for the British pound and Indian rupee.

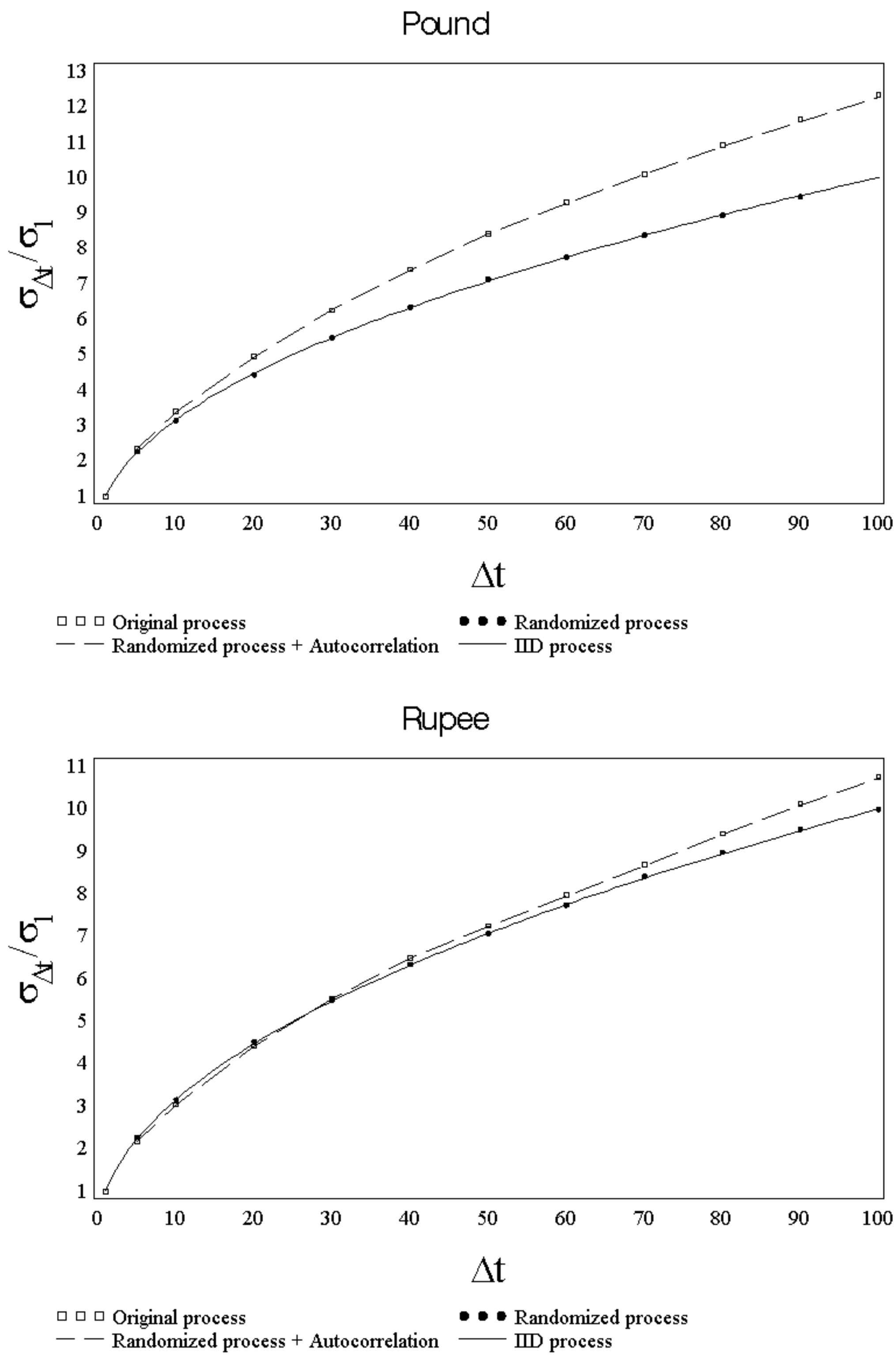


Fig. 8. Effect of the autocorrelation for the British pound and Indian rupee.

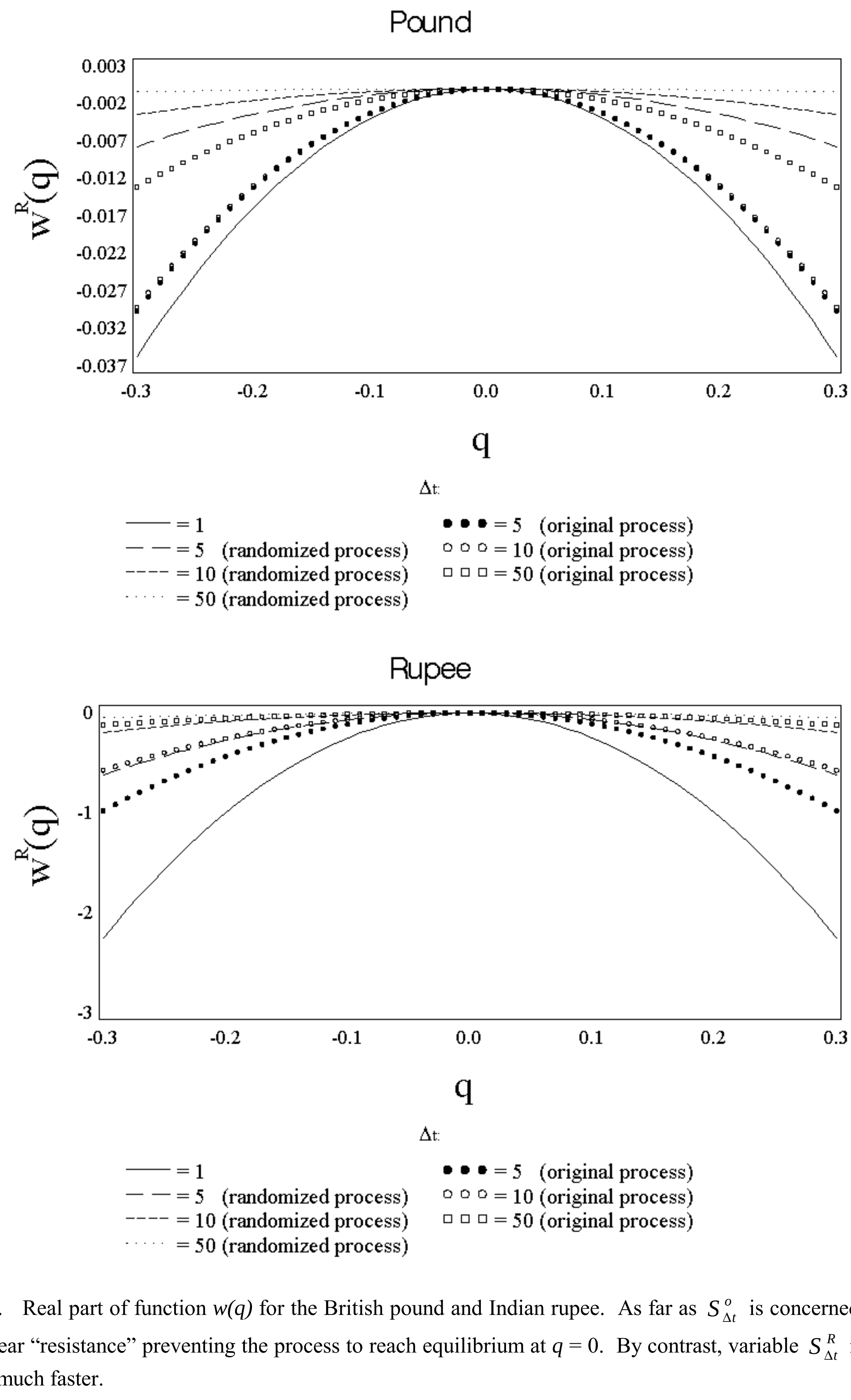


Fig. 9. Real part of function *w(q)* for the British pound and Indian rupee. As far as $S^{o}_{\Delta t}$ is concerned, there is a clear "resistance" preventing the process to reach equilibrium at $q = 0$. By contrast, variable $S^{R}_{\Delta t}$ reaches $q = 0$ much faster.

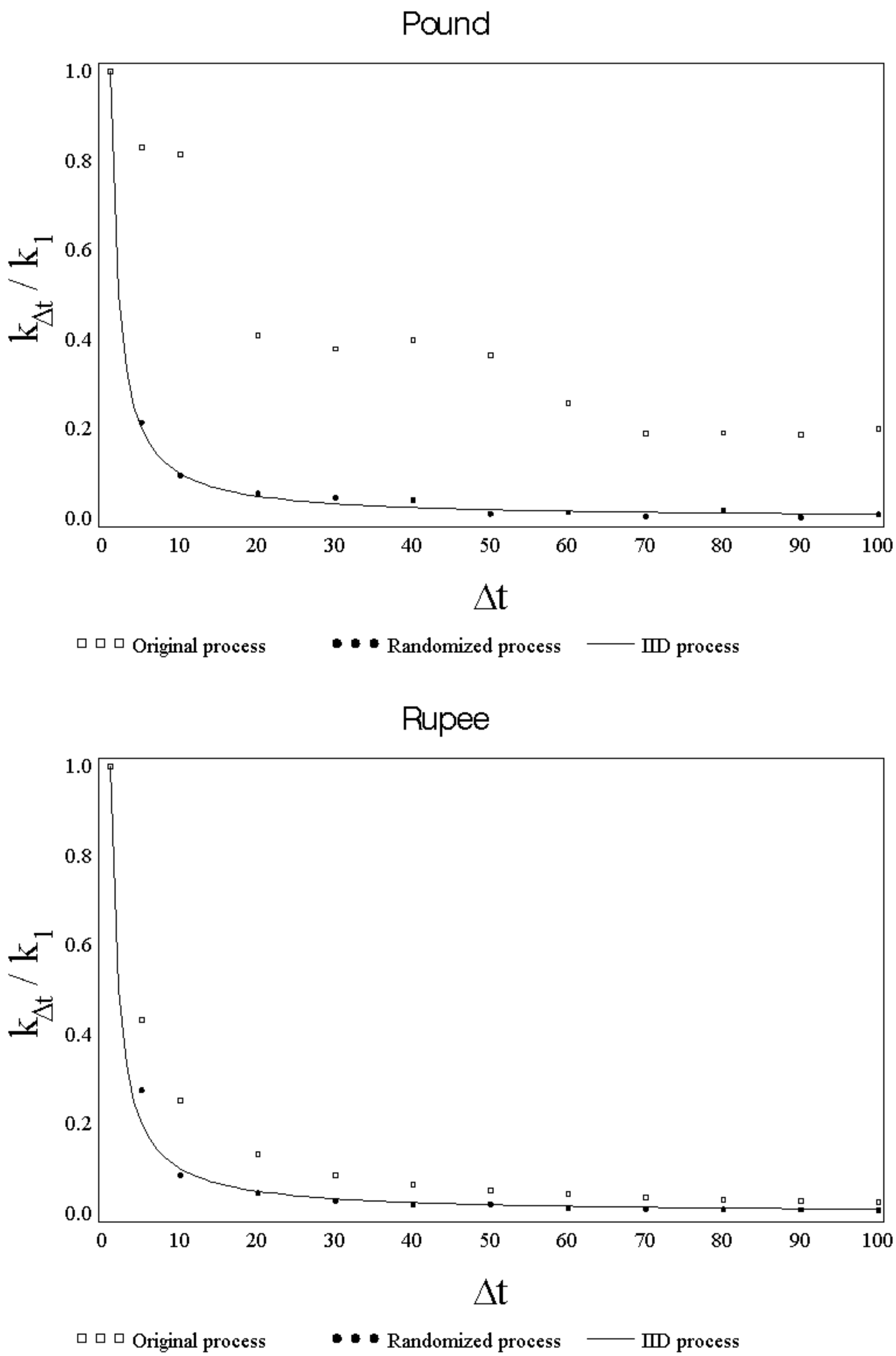


Fig. 10. Dynamic kurtosis curve of the British pound and Indian rupee.

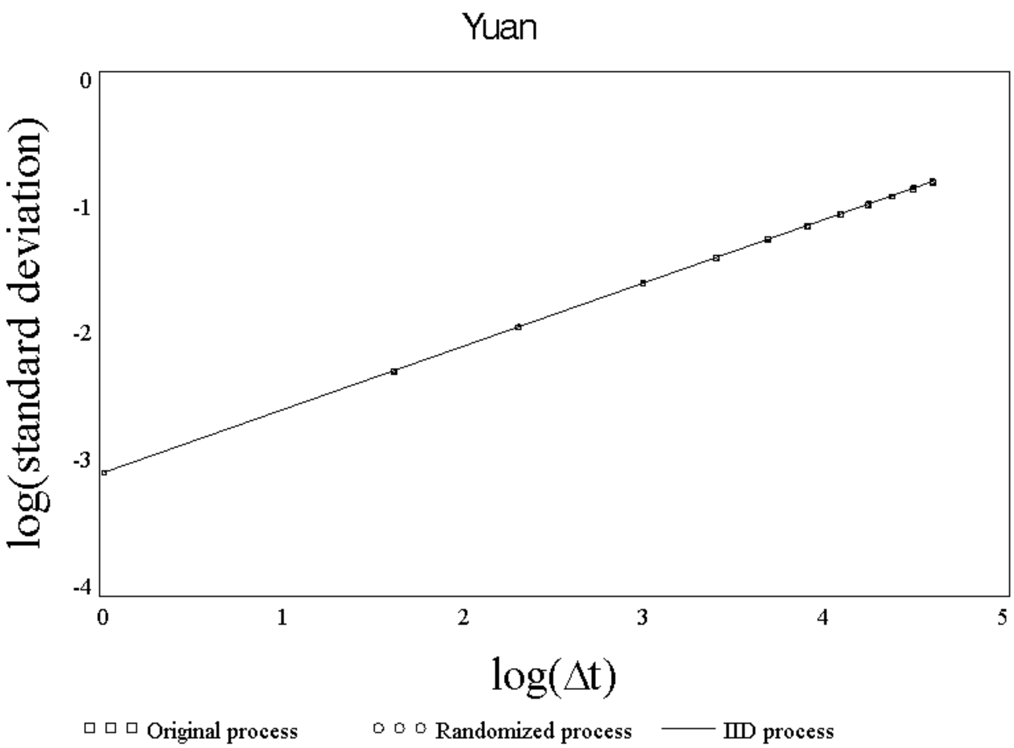
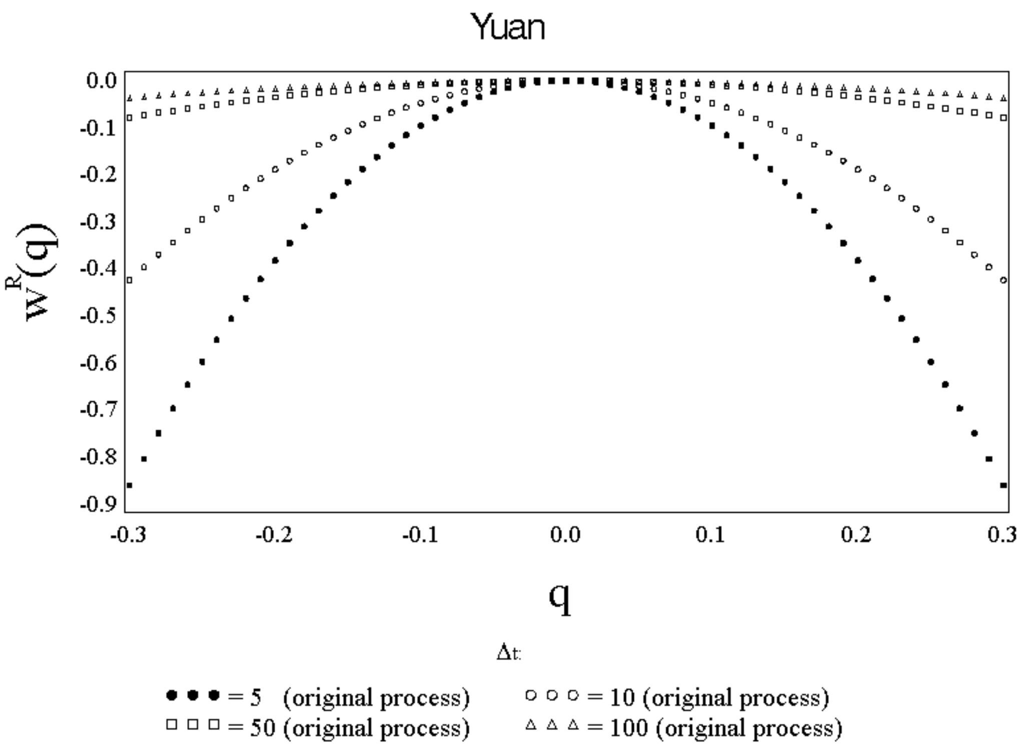


Fig. 11. Scaling in standard deviation for the processes $S^{o}_{\Delta t}$ and $S^{R}_{\Delta t}$ of the Chinese yuan. The fitting line of the original process $S^{o}_{\Delta t}$ is –3.093376 + 0.497597 log(Δt) and that of the randomized process $S^{R}_{\Delta t}$ is –3.098142 + 0.50034 log(Δt).



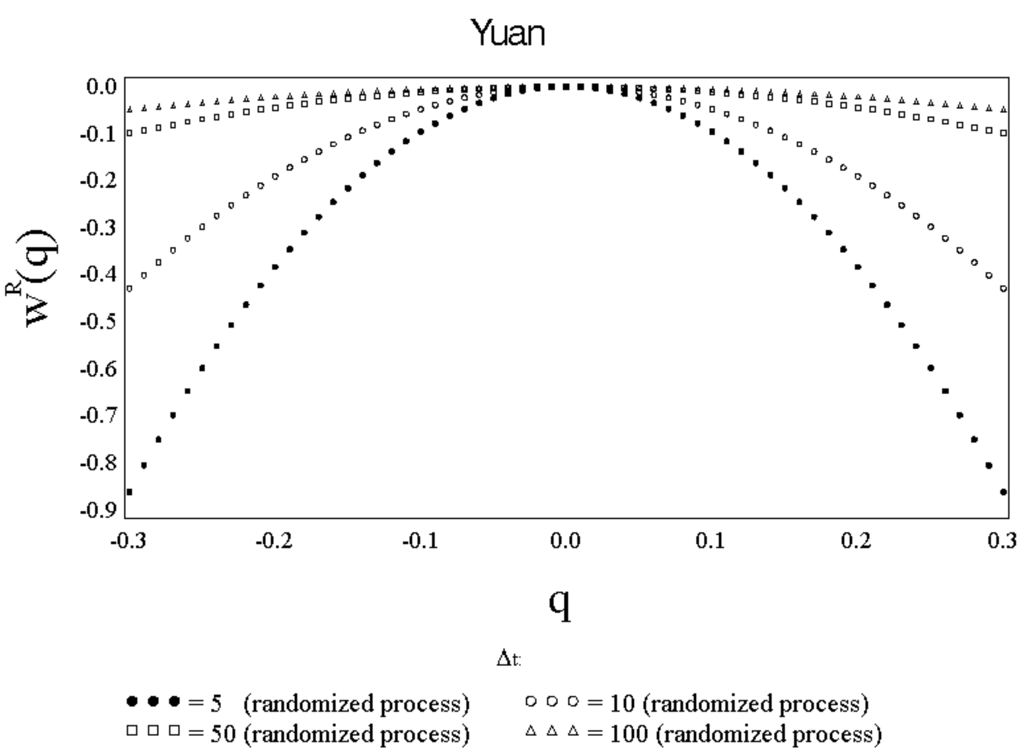


Fig. 12. Behavior of function $w(q)$ for the Chinese yuan. Since both $S^{o}_{\Delta t}$ and $S^{R}_{\Delta t}$ follow a stochastic process, $w(q)$ does not exhibit the quasi-stability necessary for the TLF to emerge.

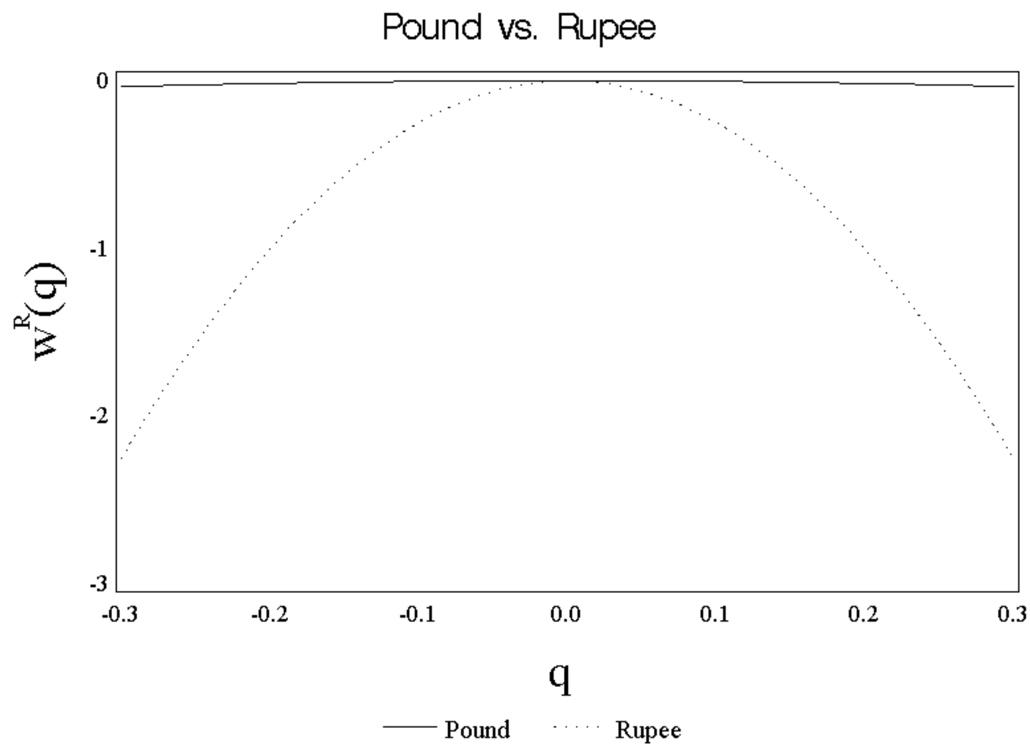


Fig. 13. Equilibrium: comparison between the British pound and Indian rupee.

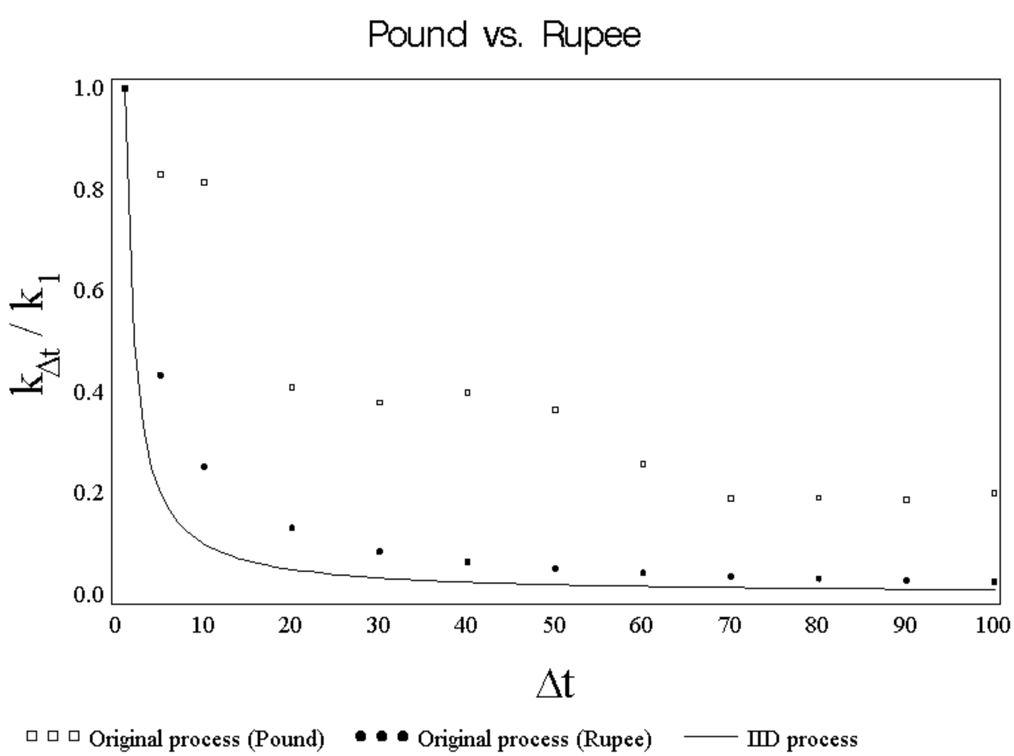


Fig. 14. Speed of convergence as measured by the kurtosis dynamic curve: comparison between the British pound and Indian rupee.