Lévy flights, autocorrelation, and slow convergence

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Abstract
Previously we have put forward that the sluggish convergence of truncated Lévy flights to a Gaussian [1] together with the scaling power laws in their probability of return to the origin [2] can be explained by autocorrelation in data [3, 4]. A purpose of this paper is to improve and enlarge the scope of such a result. The role of the autocorrelations in the convergence process as well as the problem of establishing the distance of a given distribution to the Gaussian are analyzed in greater detail. We show that whereas power laws in the second moment can still be explained by linear correlation of pairs, sluggish convergence can now emerge from nonlinear autocorrelations. Our approach is exemplified with data from the British pound-US dollar exchange rate.

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1. Introduction

We have previously suggested that the sluggish convergence associated with truncated Lévy flights (TLFs) [1] and the power laws emerging in their "probability of return to the origin" [2] can be explained on the basis of particular features of autocorrelation in data [3, 4]. Our case for the role of the autocorrelations has since been strengthened by others [5]. However, since only linear correlations of pairs have been considered in our previous work, that leaves room for examining the role of nonlinear correlations (if any) in the dynamics of a convergence process. Here we tackle this issue. And by doing so we in a sense complement our previous theory. We thus show that whereas power laws in the second moment can still be explained by linear correlation of pairs, sluggish convergence can now emerge from nonlinear autocorrelations.

As far as the convergence is concerned, we have incidentally advanced, too, a dynamic measure of the distance of a distribution from the Gaussian. Here we revisit such a result to perfect it along the lines of our broader approach.

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To illustrate our novel methodology we employ data from the British pound-US dollar.

The rest of this paper is organized as follows. Section 2 briefly revisits our previous work [3, 4]. Sections 3, 4, and 5 are devoted to explain and present novel results. Our method is then exemplified in Section 6. Finally, Section 7 concludes.

2. Quasi-stability and autocorrelation

Our previous case for the role the autocorrelations [3, 4] relies on the concept of a "quasi-stable" process, which is as follows. Take the sum of random variables x_i:

$$S_n = \sum_{i=1}^{n} x_i \tag{1}$$

Without loss of generality it can be assumed a zero mean for x_i. Lévy himself [6] shows that, for reduced variables $\bar{x}_i = x_i \big/ \sqrt{\mu_{i2}}$, the characteristic function (CF) $\psi(z)$ of a process with finite second moment can be written as

$$\overline{\psi}(z) = e^{-q^2(1+w(z))/2} \tag{2}$$

where $w(0) = 0$. The probability density function (PDF) of the distribution is given by

$$\begin{aligned} f(x) &= f_{even}(x) + f_{odd}(x) = \\ &\frac{1}{2\pi}\int_{-\infty}^{\infty} \psi_R(z)\cos(xz)dz + \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi_I(z)\sin(xz)dz \end{aligned} \tag{3}$$

with $\psi(z) = \psi_R + I\psi_I$. If we further assume that $w(z) = w_R + Iw_I$ it can be shown that

$$\begin{aligned} f(x) &= f_{even}(x) + f_{odd}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\left(z^2(1+w_R(z))/2\right)}\cos\left(z^2 w_I(z)/2\right)\cos(xz)dz + \\ &\frac{-1}{2\pi}\int_{-\infty}^{\infty} e^{-\left(z^2(1+w_R(z))/2\right)}\sin\left(z^2 w_I(z)/2\right)\sin(xz)dz \end{aligned} \tag{4}$$

Note that w_R and w_I appear in $f_{even}(x)$ and $f_{odd}(x)$.

Definition 1. A *stable process* occurs in interval $[n_1, n_2]$ if

$$\Omega_n(z) = constant, \quad \forall n_1 \le n \le n_2 \tag{5}$$

and $\overline{\Psi}_n(z) = e^{-q^2(1+\Omega_n(z))/2}$ is the CF of S_n. Definition 1 cannot turn the PDFs stable. But Definition 1 makes the CF constant in the interval $[n_1, n_2]$. For such a class of processes it can be shown that

$$P(0) = \frac{constant'}{\nu_n} \quad (6)$$

where $P(0)$ is the probability of return to the origin, and ν_n is the standard deviation of S_n. If ν_n scales as a power law of type

$$\nu_n \propto n^{1/\alpha}, \qquad n_1 \le n \le n_2 \quad (7)$$

then we have

$$P(0) = \frac{constant''}{n^{1/\alpha}} \quad (8)$$

This is a scaling property observed in TLFs, which holds for a finite time window until "termalization" ($\alpha = 2$) takes place at a ultraslow pace, as shown by Mantegna and Stanley [1].

For practical purposes a *quasi-stable* process, where $\Omega_n(z)$ is almost constant in a given time window, sounds more appropriate. For quasi-stable processes, power law (8) holds temporarily.

For independent and identically distributed (IID) variables, $\alpha = 2$ in (4). In fact, we have

$$\begin{aligned} {\nu_{S_n}}^2 &= n\nu^2 + 2\nu^2\left(\sum_{i,j=1}^{n}(<x_i x_j> - <x_i><x_j>)/\nu^2\right) \\ \nu_{S_n} &= \nu\sqrt{n + 2\sum_{\substack{i,j \\ i\neq j}}^{n} corr(x_i, x_j)} = \nu\sqrt{n + \sum_{i,j}^{n} corr(x_i, x_j)} \end{aligned} \quad (9)$$

where ν is the standard deviation of x_i. Thus the TLF can be linked to both quasi-stability and scaling in volatility. The latter can be explained on the basis of linear correlation of pairs [3, 4]. Next sections will examine the role of nonlinear autocorrelations for the sluggish convergence.

3. The role of autocorrelations in the sum of stochastic variables

Let us write the statistical moments of x_i and S_n as

$$\mu_{ip} = <x_i^{\,p}>, \forall i, p \in N \quad (10)$$

and

$$\nu_{np} = <S_n^{\,p}>, \forall n, p \in N \quad (11)$$

respectively. Moreover

$$\sigma_{np} = \mu_{1p} + \mu_{2p} + \cdots + \mu_{np} = < x_1^p > + \cdots + < x_n^p > \tag{12}$$

As observed, Lévy proved [6] that, for a random variable x_i with a finite second moment, the CF of the reduced variable $\bar{x}_i = x_i \big/ \sqrt{\mu_{i2}}$ satisfies

$$\overline{\psi}_i(z) = e^{-z^2(1+w(z))/2}, w(0) = 0 \tag{13}$$

Writing $\psi_i(z)$ for the CF of x_i yields $\overline{\psi}_i(z) = \psi_i\left(z/\sqrt{\mu_{i2}}\right)$. The CF of x_i and S_n can then be written as

$$\psi_i(z) = < e^{Ix_i z} > = e^{-\mu_{i2} z^2 (1+w_i(\sqrt{\mu_{i2}} z))/2} \tag{14}$$

and

$$\Psi_n(z) = < e^{IS_n z} > = e^{-\nu_{n2} z^2 (1+\Omega_n(\sqrt{\nu_{n2}} z))/2} \tag{15}$$

respectively.

The existence of a CF for the sum variable when $n \to \infty$ is guaranteed by Lévy's continuity theorem. For independent variables it holds true that

$$\Psi_n(z) = \psi_1(z) \cdots \psi_n(z) \tag{16}$$

Eq. (16) does not hold for autocorrelated processes, though. For the latter we propose that

$$\Psi_n(z) = C_n(z) \psi_1(z) \cdots \psi_n(z) \tag{17}$$

where function $C_n(z)$ allows one to get a measure of independence between the variables x_i. These are independent at the borderline case when $C_n(z) = 1$; otherwise, autocorrelations are present. Section 4 will evaluate $C_n(z)$ to take the measurement of the degree of dependence into account.

Expressions (15) can be expanded in series to give

$$\psi_i(z) = 1 + \frac{I^2}{2!} \mu_{i2} z^2 + \frac{I^3}{3!} \mu_{i3} z^3 + \frac{I^4}{4!} \mu_{i4} z^4 + o(z^4) \tag{18}$$

and

$$\Psi_n(z) = 1 + \frac{I^2}{2!}\nu_{n2}z^2 + \frac{I^3}{3!}\nu_{n3}z^3 + \frac{I^4}{4!}\nu_{n4}z^4 + o(z^4) \tag{19}$$

Now let us suppose that $C_n(z)$ allows a series expansion in *z*, i.e.

$$C_n(z) = 1 + C_{n2}z^2 + C_{n3}z^3 + C_{n4}z^4 + o(z^4) \tag{20}$$

By employing Eq. (18) it can be shown that

$$\psi_1(z)\cdots\psi_n(z) = 1 + \frac{I^2}{2!}\sigma_{n2}z^2 + \frac{I^3}{3!}\sigma_{n3}z^3 + I^4\left(\frac{\sigma_{n4}}{4!} + \frac{\gamma_n}{2!2!}\right)z^4 + o(z^4) \tag{21}$$

with

$$\gamma_n = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\mu_{i2}\mu_{j2} = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} < x_i^2 >< x_j^2 > \tag{22}$$

Substituting (19), (12), and (21) into (9) yields

$$\begin{aligned} &1 + \frac{I^2}{2!}\nu_{n2}z^2 + \frac{I^3}{3!}\nu_{n3}z^3 + \frac{I^4}{4!}\nu_{n4}z^4 + o(z^4) = \\ &\left(1 + C_{n2}z^2 + C_{n3}z^3 + C_{n4}z^4 + o(z^4)\right)\times \\ &\times\left(1 + \frac{I^2}{2!}\sigma_{n2}z^2 + \frac{I^3}{3!}\sigma_{n3}z^3 + I^4\left(\frac{\sigma_{n4}}{4!} + \frac{\gamma_n}{2!2!}\right)z^4 + o(z^4)\right) \end{aligned} \tag{23}$$

Comparing equal order terms produces

$$C_{n2} = -\frac{1}{2}\left(\nu_{n2} - \sigma_{n2}\right) \tag{24}$$

$$C_{n3} = -\frac{I}{3!}\left(\nu_{n3} - \sigma_{n3}\right) \tag{25}$$

and

$$C_{n4} = \frac{1}{4!}\left(\nu_{n4} - \sigma_{n4}\right) - \frac{1}{2!2!}\left(\sigma_{n2}\left(\nu_{n2} - \sigma_{n2}\right) + \gamma_n\right) \tag{26}$$

By taking (20) into account it can be assumed without loss of generality that

$$C_n(z) = e^{-\frac{z^2}{2}\left(-2C_{n2} + W_n(z)\right)} \tag{27}$$

where $W_n(z)$ allows a series expansion such that

$$W_n(z) = IW_{n1}z + W_{n2}z^2 + o(z^2) \tag{28}$$

Then it can be shown from (24)–(26) that

$$W_{n1} = \frac{1}{3}\left(\nu_{n3} - \sigma_{n3}\right) \tag{29}$$

and

$$W_{n2} = \frac{1}{4}\left(\nu_{n2} - \sigma_{n2}\right)^2 - 2C_{n4} \tag{30}$$

After taking Eq. (14) into account, Eq. (21) can be rewritten as

$$\psi_1(z)\cdots\psi_n(z) = \prod_{i=1}^{n} e^{-\mu_{i2}z^2\left(1+w_i\left(\mu_{i2}^{1/2}z\right)\right)/2} = e^{-z^2\left(\sigma_{n2}+\sum_{i=1}^{n}\mu_{i2}w_i(\mu_{i2}^{1/2}z)\right)/2} \tag{31}$$

After considering Eqs. (28) and (31), the CF in (17) becomes

$$\begin{aligned} \Psi_n(z) &= e^{-z^2\left(\sigma_{n2}+\sum_{i=1}^{n}\mu_{i2}w_i(\mu_{i2}^{1/2}z)\right)/2} e^{-z^2(\nu_{n2}-\mu_{n2}+W_n(z))/2} \Rightarrow \\ \Psi_n(z) &= e^{-z^2\left(\nu_{n2}+\sum_{i=1}^{n}\mu_{i2}w_i(\mu_{i2}^{1/2}z)+W_n(z)\right)/2} \end{aligned} \tag{32}$$

And for the reduced variables one has

$$\overline{\Psi}_n(z) = \Psi\left(z/\sqrt{\nu_{n2}}\right) = e^{-z^2(1+\Omega_n(z))/2} \tag{33}$$

We now define

$$\Omega_n(z) = \Omega_n^{(1)}(z) + \Omega_n^{(2)}(z) \tag{34}$$

and use Eq. (32) to get

$$\Omega_n^{(1)}(z) = \frac{1}{\nu_{n2}}\sum_{i=1}^{n}\mu_{i2}w_i\left(z\sqrt{\mu_{i2}}/\sqrt{\nu_{n2}}\right) \tag{35}$$

and

$$\Omega_n^{(2)}(z) = \frac{1}{\nu_{n2}} W_n\left(z / \sqrt{\nu_{n2}}\right) \tag{36}$$

Function $\Omega_n^{(1)}(z)$ matches the one for uncorrelated series, i.e. as $n \to \infty$ it approaches $w(0) = 0$. And term $\Omega_n^{(2)}(z)$ is related to the existence of autocorrelations. It is key to understanding the ultraslow convergence. The term gives the CF of the sum variable, which in turn can be used to obtain its PDF as $n \to \infty$. This result in a sense generalizes the central limit theorem for autocorrelated processes to which Eq. (17) holds.

4. Autocorrelation and convergence

Now we look for an expression to $\Omega_n^{(2)}(z)$ containing uniquely the statistical moments of a distribution. That renders it suitable for practical applications. Let us first define a nonlinear correlation term:

$$< p_1 p_2 \cdots p_k >_n = \sum_{i_1 \cdots i_k = 1}^{n} \left(< x_{i_i}^{p_1} \cdots x_{i_k}^{p_k} > - < x_{i_i}^{p_1} > \cdots < x_{i_k}^{p_k} > \right) \tag{37}$$

-where $p_1 p_2 \cdots p_k$ are positive integers, and $i_1 \neq i_2 \neq \cdots \neq i_k$. For example, if we employ <11> in Eq. (37) we obtain the usual linear correlation of pairs, i.e.

$$< 11 >_n = \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \left(< x_i x_j > - < x_i >< x_j > \right) = \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \left(< x_i x_j > \right) = \nu_{n2} - \sigma_{n2} \tag{38}$$

Eq. (37) gives our definition of nonlinear autocorrelations. Our task is to evaluate the contribution of such nonlinear terms in the process of convergence. By using (28), (29), and (30) it can be shown that

$$\Omega_n^{(2)} = \Omega_{n1}^{(2)} zI + \Omega_{n2}^{(2)} z^2 \tag{39}$$

$$\Omega_{n1}^{(2)} = \frac{1}{3} \frac{\nu_{n3} - \sigma_{n3}}{\nu_{n2}^{3/2}} = \frac{< 111 >_n + 3 < 12 >_n}{\nu_{n2}^{3/2}} \tag{40}$$

and

$$\Omega_{n2}^{(2)} = -\frac{1}{12} \frac{\nu_{n4} - \sigma_{n4} - 6\gamma_n}{\nu_{n2}^2} + \frac{1}{4}\left(1 - \frac{\sigma_{n2}^2}{\nu_{n2}^2}\right)$$
$$= (-1/12)(< 1111 >_n + 6 < 112 >_n + 4 < 13 >_n + 3 < 22 >) / \nu_{n2}^2 + R_n \tag{41}$$

where $\Omega_{n1}^{(2)}$ and $\Omega_{n2}^{(2)}$ are functions of third- and fourth order correlations respectively.

Now let us turn to analyze the role of the linear correlations in convergence to the Gaussian regime. First note that

$$R_n = \frac{1}{4}\left(1 - \frac{\sigma_{n2}^2}{\nu_{n2}^2}\right) \tag{42}$$

depends only on ν_{n2} and μ_{n2}, which are related by Eq. (38). As $\sigma_{n2}/\mu_{n2} \to 1$, R_n approaches zero; according to Eq. (38) that is possible only if $<11>_n / \sigma_{np} \to 0$, i.e. if σ_{n2} dominates the sum of all linear correlations in the process. Likewise we are sure that R_n cannot approach zero if term $<11>_n$ is relevant.

As an illustration, consider identically distributed variables for which

$$\begin{aligned} &\mu_{12} = \cdots = \mu_{n2} = m^2 \\ &\sigma_{n2} = nm^2 \\ &\nu_{n2} = \left(n + \frac{<11>_n}{m^2}\right)m^2 \end{aligned} \tag{43}$$

The sum of all linear correlations in the variables x_i is given by $<11>_n / m^2$. As shown elsewhere [3], the presence of such correlations is responsible for the emergence of a power law in the cumulative dispersion S_n, i.e.

$$\sqrt{\nu_{n2}} \to mn^{\alpha}, \alpha > 1/2 \tag{44}$$

As a result, one has, too, that

$$R_n \to \frac{1}{4}\left(1 - \frac{1}{n^{4(\alpha-1/2)}}\right) \tag{45}$$

For $n \to \infty \Rightarrow R_n \to 1/4$, and this prevents termalization to take place. For actual data, Eq. (44) is expected to hold temporarily and as an approximation; so $R_n \to constant \neq 1/4$. Thus the linear correlations can cause delays in the convergence process; and that holds true even for short-range autocorrelated processes. Nevertheless if $R_n \to 0$ then the nonlinear autocorrelations matter. And here the measure of dependence can be evaluated from Eqs. (40) and (41). By considering these equations, next section will tackle the problem of the distance to the Gaussian regime.

5. Distance to the Gaussian regime

Now we turn to the problem of how to measure the distance of a process which currently stands away from its ultimate Gaussian equilibrium. By using Eqs. (2) and (18), we write $w(z) = w_R + Iw_I$. With the help of some algebra it can be shown that

$$\begin{cases} w_R = -\frac{1}{12}(\mu^4 - 3)z^2 - (\frac{\mu^4}{24} + \frac{\mu^3}{36} - \frac{1}{12} - \frac{\mu^6}{360})z^4 + ... \\ w_R = -\frac{1}{12}(K)z^2 + O(z^4) \end{cases} \tag{46}$$

and

$$\begin{cases} w_I = \frac{\mu^3}{3}z + (\frac{1}{6}\mu^3 - \frac{\mu^5}{60})z^3 + \frac{1}{12}(\mu^3 - \frac{\mu^5}{10} - \frac{\mu^3\mu^4}{6})z^5 + O(z^7) \\ w_I = \frac{1}{3}(Sk)z + \frac{1}{6}(Sk)z^3 + O(z^5) \end{cases} \tag{47}$$

where

$$\mu^{\Delta t} = \frac{1}{N}\sum_{i=1}^{N}\left[\frac{x_i - <x>}{\sigma}\right]^{\Delta t} \tag{48}$$

is the n^{th} order momentum,

$$K = \frac{1}{N}\sum_{i=1}^{N}[(x_i - <x>)/\sigma]^4 - 3 = \mu^4 - 3 \tag{49}$$

is the kurtosis, and

$$Sk = \frac{1}{N}\sum_{i=1}^{N}\left[\frac{x_i - <x>}{\sigma}\right]^3 \tag{50}$$

is the skewness.

We have proposed [3] that the norm of $w(z)$ gives a good measure of the distance of a given PDF to the Gaussian one, where $w(z) = 0$. For a given δ, the distance between two distributions f and g of a *reduced normal family* is given by

$$D(f,g) = \int_{-\delta}^{\delta}\sqrt{\left(w_R^f(z) - w_R^g(z)\right)^2 + \left(w_I^f(z) - w_I^g(z)\right)^2}\,dz \tag{51}$$

So if g is the reduced, normal distribution, the distance of a given f to the Gaussian is

$$D(f, Gauss) = \int_{-\delta}^{\delta}\sqrt{w_R^f(z)^2 + w_I^f(z)^2}\,dz \tag{52}$$

Terms w_R and w_I are constrained to some real value in a finite time window for systems with sluggish convergence, such as either our class of quasi-stable processes or the TLF. Unlike an IID process, here the kurtosis K and the skewness Sk cannot decay to zero following curves $1/n$ and $1/n^{1/2}$ respectively. For the sum variable, after writing

$$\overline{\Psi}_n(z) = \Psi\left(z/\sqrt{\nu_{n2}}\right) = e^{-z^2(1+\Omega_n(z))/2} \Rightarrow$$

$$\begin{cases} \Omega_{nR}(z) = -\dfrac{1}{12}(K_n)z^2 + O(z^4), \\ \Omega_{nI}(z) = \dfrac{1}{3}(Sk_n)z + \dfrac{1}{6}(Sk_n)z^3 + O(z^5) \end{cases} \tag{53}$$

we can show that

$$\frac{1}{3}Sk_n = \sum_{i=1}^{N}\frac{\mu_{i2}^{3/2}\mu_{i3}}{3\nu_{n2}^{3/2}} + \frac{<111>_n + 3<12>_n}{\nu_{n2}^{3/2}} \equiv \frac{1}{3}Sk_n^0 + Sk_n^1 \tag{54}$$

Term Sk_n^0 is the skewness of S_n for an IID process. Correlations of third order can then be responsible for preventing w_I to reach zero through Sk_n^1. It can be also shown, after considering Eqs. (33) and (34), that

$$\begin{aligned} \frac{1}{12}K_n &= \sum_{i=1}^{N}\frac{\mu_{i2}^2\mu_{i4}}{12\nu_{n2}^2} + \frac{1}{4}\left(1 - \frac{\sigma_{n2}^2}{\nu_{n2}^2}\right) - \\ \frac{1}{12}&\frac{<1111>_n + 6<112>_n + 4<13>_n + 3<22>_n}{\nu_{n2}^2} \equiv \frac{1}{12}K_n^0 + K_n^1 + K_n^2 \end{aligned} \tag{55}$$

Term K_n^0 is that of an IID process. Term K_n^1 $(=R_n)$ contains linear correlation of pairs. But correlations of fourth order appear in K_n^2. Both linear and nonlinear correlations are critical for analysis of w_R. Although linear correlations play a key role in convergence of the distribution, it is still necessary to take the nonlinear autocorrelations into account to fully characterize the process.

It is worth noting a similarity between Eqs. (53)–(54) and (40)–(42). The latter were obtained from the assumption that dependence is fully accounted by the presence of $C_n(z)$ whereas Eqs. (53)–(54) were obtained without any prior assumption. From both approaches we come to the conclusion that nonlinear autocorrelations play an important role in the sum of stochastic variables. Although that is arguably well known in literature, our novel methodology presents formulas that allow one to evaluate the contributions of the nonlinear terms explicitly. What is more, our formulas for statistical momenta are quantities easily obtained with the help of a computer.

6. Applications

Now we exemplify our approach with data coming from daily variations of the British pound against the US dollar. Our data set contains 8213 data points, covering the time period from 1 April 1971 to 26 September 2003. As usual, we take returns Z rather than raw data as our stochastic variable, i.e.

$$Z_{\Delta t}(t) = Y(t + \Delta t) - Y(t) \tag{56}$$

where $Y(t)$ is a rate at day t. Note that $Z_{\Delta t}(t) \equiv S_n$ and $\Delta t \equiv n$.

Fig. 1 displays $\nu_n^{1/2}$ against n. Curve $\sqrt{n}$ of an IID process is also shown for comparison. Scaling power law

$$\begin{aligned} \sigma^o_{S_n} &= 0.0097055251\, n^{0.55091} \\ \frac{1}{\alpha} &= 0.55091 \Rightarrow \alpha = 1.815178 \end{aligned} \tag{57}$$

emerges. The autocorrelation function is presented in Fig. 2. It is interesting to compare the results in Figs. 1 and 2 with those of a variable built from a random aggregation of the original data, say S_n^R [3]. For such a variable, curve ν_n against n is expected to follow $\sqrt{n}$, as indeed is the case (not shown).

When one looks at the autocorrelation function of S_n^R (Fig. 3), one cannot tell the difference between the curve and that of Fig. 2. Both are "short-range" correlated. But something else in the autocorrelation function of Fig. 2 is still responsible for the appearance of scaling law (57).

Previously we have shown [3, 4] that S_n for the pound is sluggish, and presents a ultraslow convergence together with other properties compatible with the presence of a TLF. Are the linear correlations of pairs enough to explain the slow speed of convergence? Definitely not, as can be appreciated from the results in Sections 4 and 5.

Fig. 4 shows the curve of Eq. (53) for $\delta = 1$. It can be seen that the function is somewhat constrained to some real value which prevents termalization ($w(0) = 0$) to take place.

Figs. 5, 6, and 7 display a distance compared to $D_R \equiv \int_{-\delta}^{\delta} \sqrt{w_R^f(z)^2}$ and $D_I \equiv \int_{-\delta}^{\delta} \sqrt{w_I^f(z)^2}$. For the pound-dollar rate, D_I is the main responsible for saturation of $w(z)$ at a nonzero, real value. Also, from Eq. (4) one can see that either $w_R \neq 0$ or $w_I \neq 0$ is sufficient for termalization not to emerge.

Fig. 6 displays D, D_R, and D_I jointly. The leading terms of w_R and w_I, i.e. K_n and Sk_n respectively are presented in Figs. 7 and 14. These figures also show curves $1/n$ (for the kurtosis) and $1/\sqrt{n}$ (for the skewness) followed by an IID process, which gives the behavior of K_n^0 and Sk_n^0.

Figs. 8 to 13 present the behavior of each term in the kurtosis. They show R_n, which equals K_n^1 and accounts for the contribution of the linear autocorrelations in the dynamics of the process, together with K_n^2.

Finally, Figs. 15 and 16 present $\Omega_{n1}^{(2)}$ and $\Omega_{n2}^{(2)}$. From Eq. (36) it can be seen that when $\Omega_n^{(2)} \to \varepsilon \neq 0$ the limit distribution is not a Gaussian. From Fig. 15 one cannot say for sure that this is the case of our example, because we have stopped at n = 500. However, the fact that $\Omega_n^{(2)}$ is always different from zero in that time window do provide an explanation for the slow convergence in terms of nonlinear autocorrelations in the behavior of the kurtosis and skewness.

7. Conclusion

This paper revisits our previous result that the slow convergence associated with truncated Lévy flights and the scaling power laws emerging in their "probability of return to the origin" can be explained by autocorrelation in data [3, 4]. Our case for the role of the autocorrelations has since been strengthened by others [5]. However, here we enlarge the scope of such a result. And the pound-dollar rate is taken to illustrate our discussion.

Since only linear correlations of pairs are considered in our previous work, here we tackle the issue of the role of nonlinear autocorrelations in the dynamics of a convergence process. We thus show that, whereas power laws in the second moment can still be accounted for by linear correlation of pairs, sluggish convergence can now emerge from the nonlinear autocorrelations.

Indeed, the standard deviation exhibits power law scaling for a finite time window, which ranges from 1 to ≈ 100; and that occurs thanks to some particular features of the linear correlation of pairs (though it is not "short-range" correlated).

The process studied also presents quasi-stability, as seen from the sluggish behavior of $w(z)$. And a novelty in this paper is to show that the nonlinear autocorrelations are responsible for that sort of behavior. For instance, $w_R \neq 0$ thanks to correlations of fourth order.

As before, both quasi-stability and the power law in volatility are sufficient for the Lévy distribution to fit the modal region of a distribution [3, 4]. Here scaling breaks down at $\Delta t \approx 100$ after which the process reaches $\alpha = 2$.

Thus, whereas linear autocorrelations can explain the emergence of power law scaling in volatility, nonlinear autocorrelations are needed for one to fully characterize convergence to the Gaussian regime.

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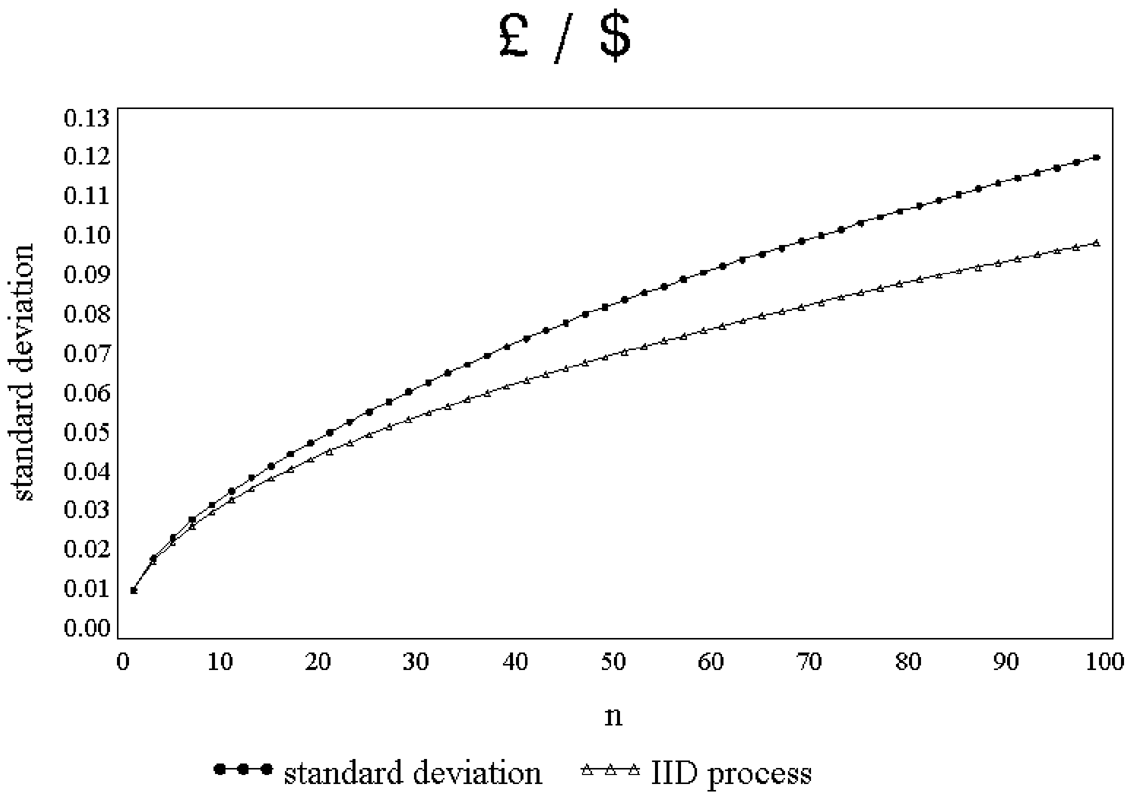

Fig. 1. Standard deviation versus *n* for the pound-dollar rate together with the curve for an IID process

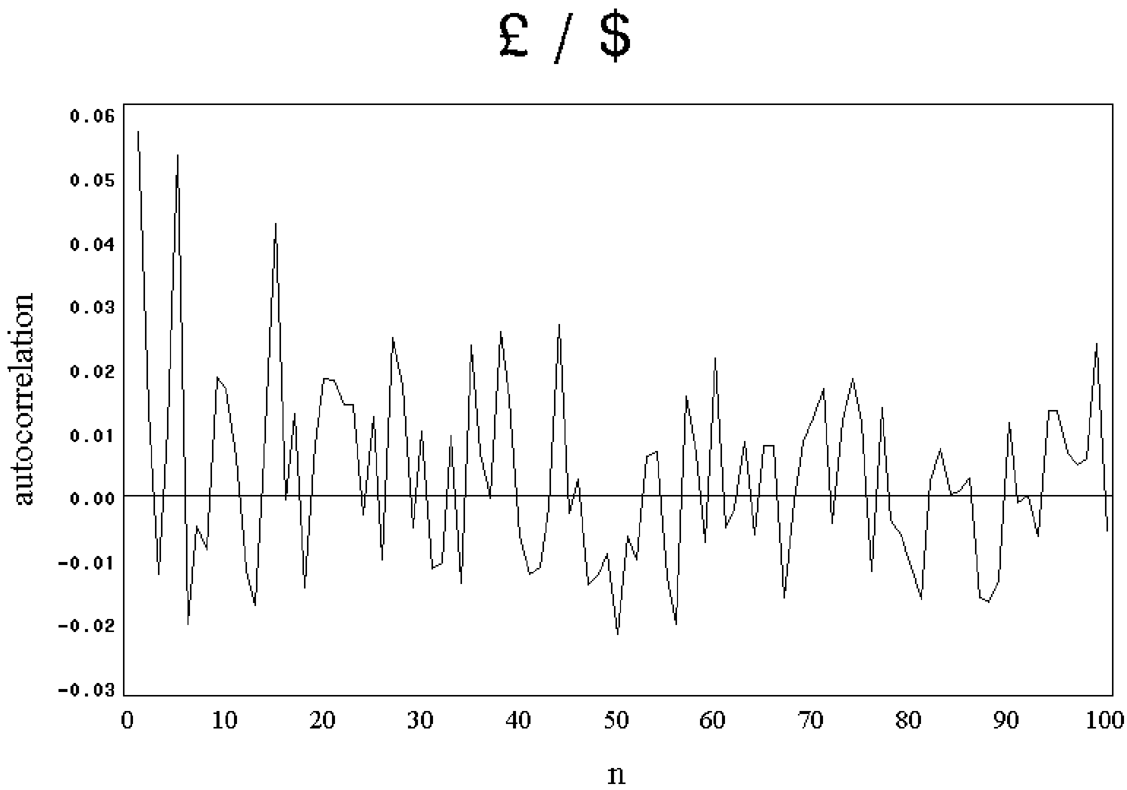

Fig. 2. Linear correlation of pairs for the pound-dollar rate

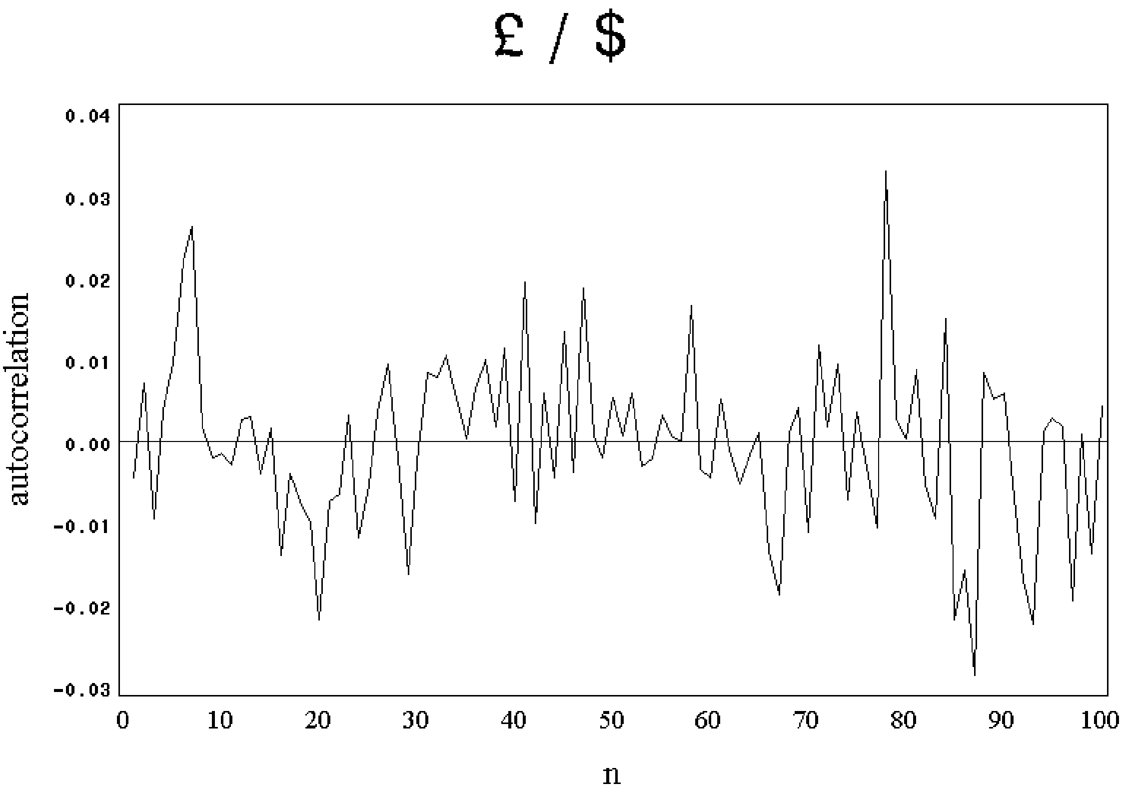

Fig. 3. Linear correlation of pairs for randomized data from the pound-dollar rate

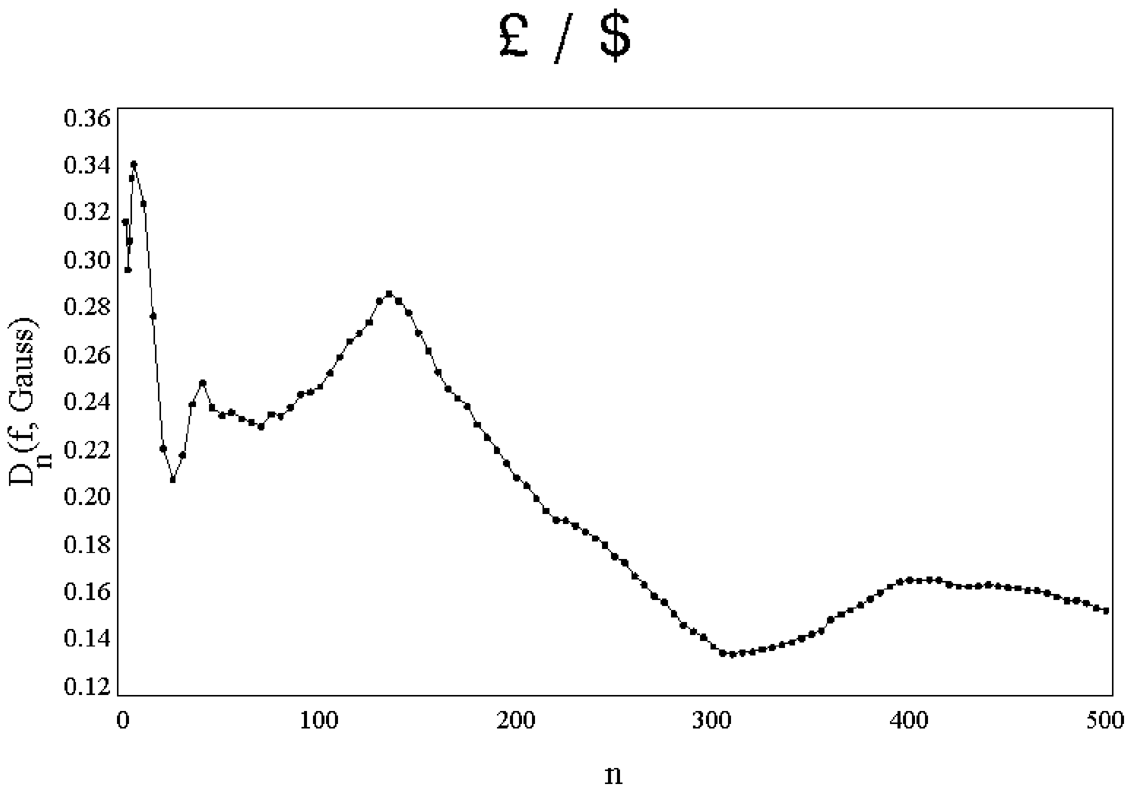

Fig. 4. Distance to the Gaussian distribution ($D(f,$ Gauss)) versus n

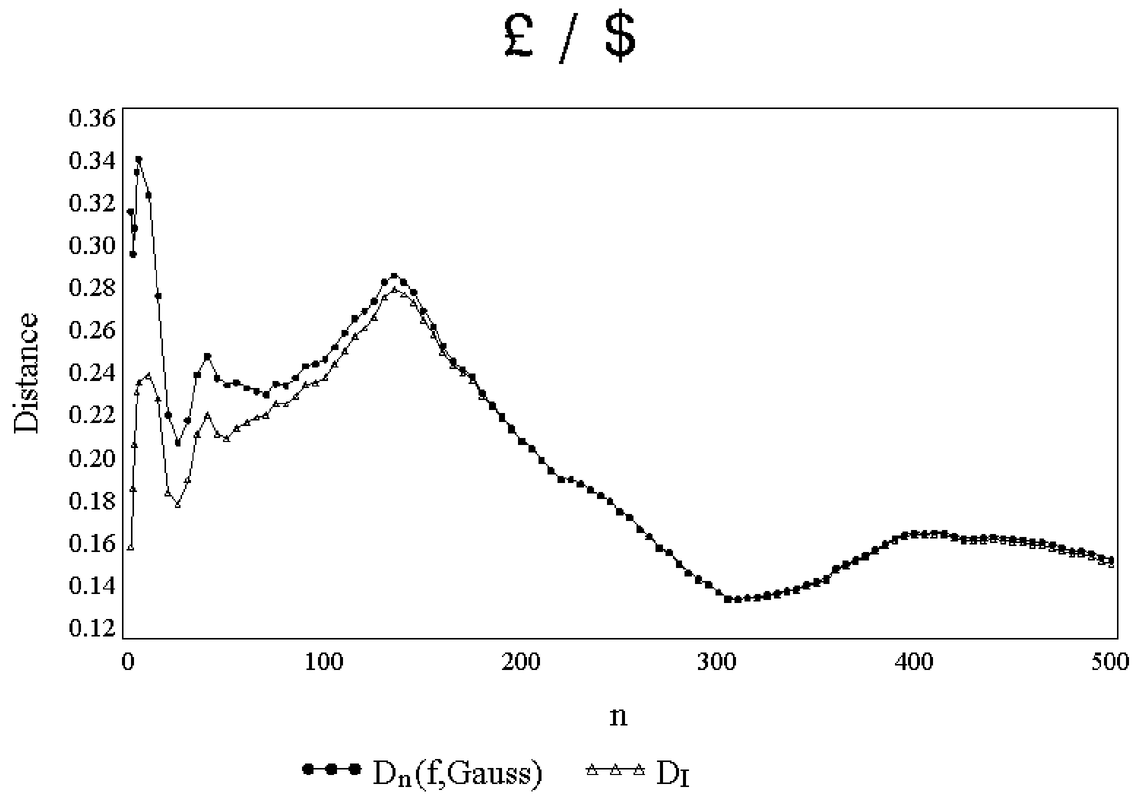

Fig. 5. Comparison between *D*(*f,* Gauss) and D_I.

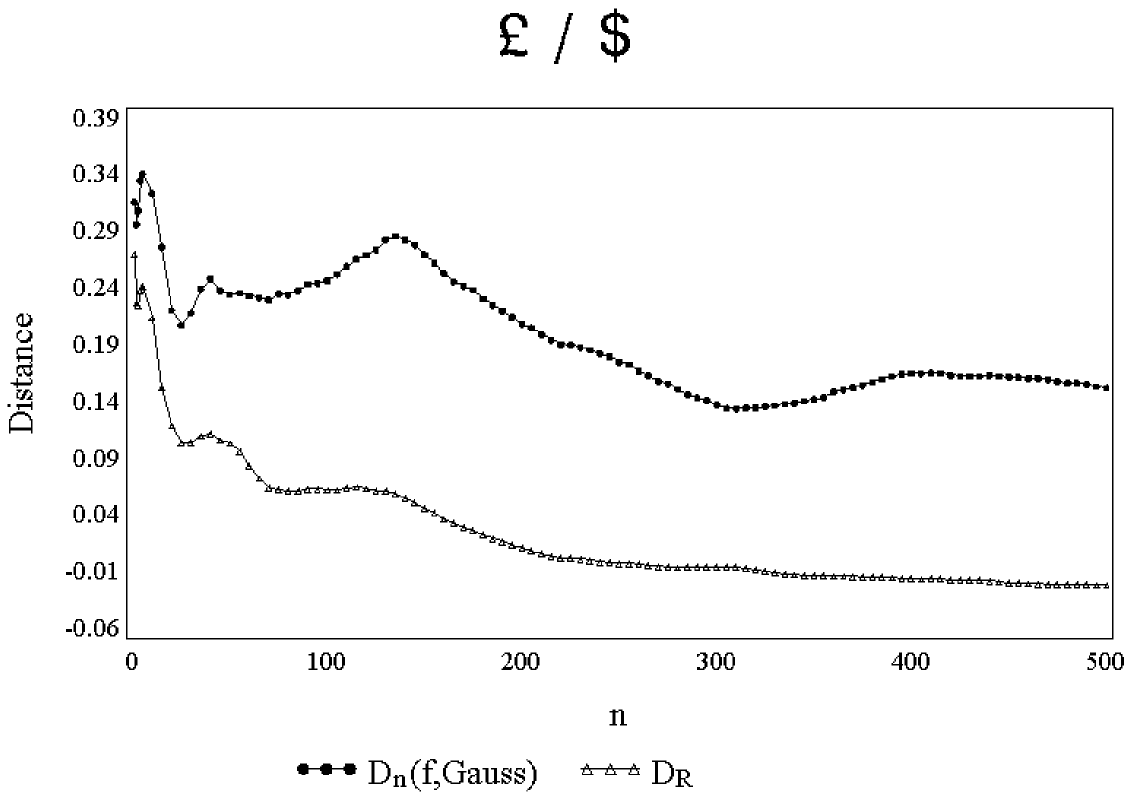

Fig. 6. Comparison between *D*(*f,* Gauss) and D_R

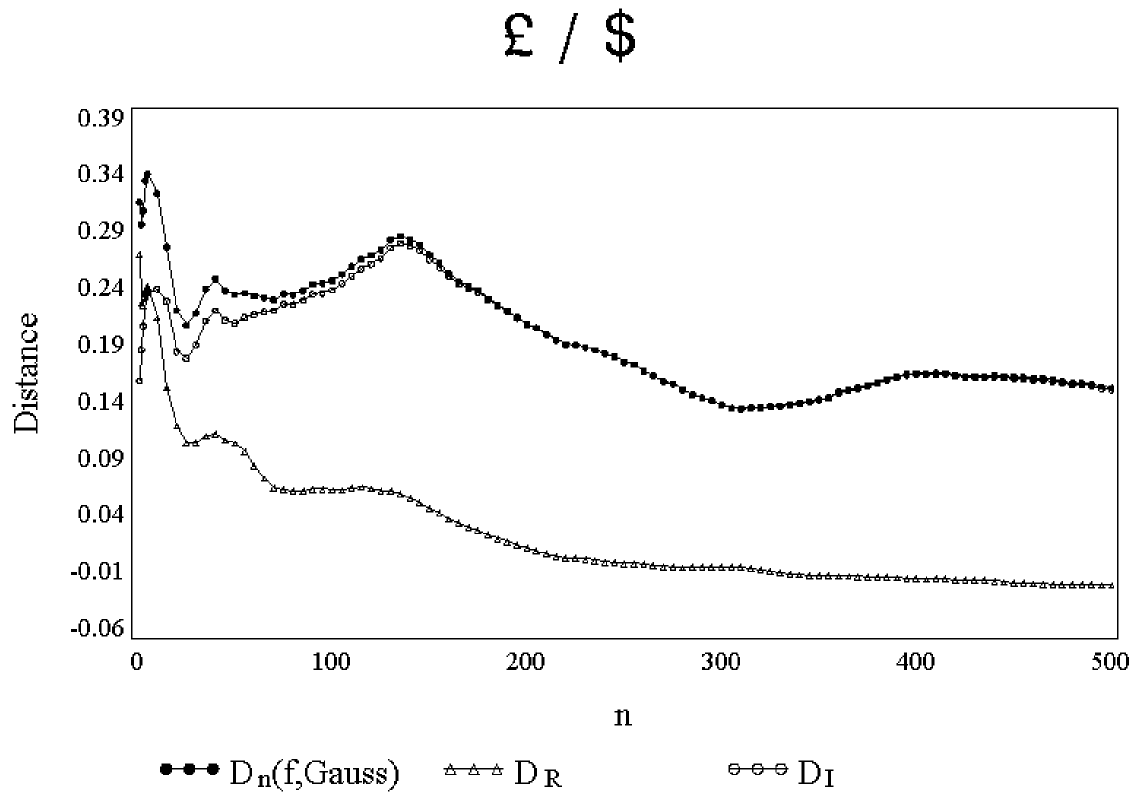

Fig. 7. Comparison between *D*(*f,* Gauss), D_R, and D_I

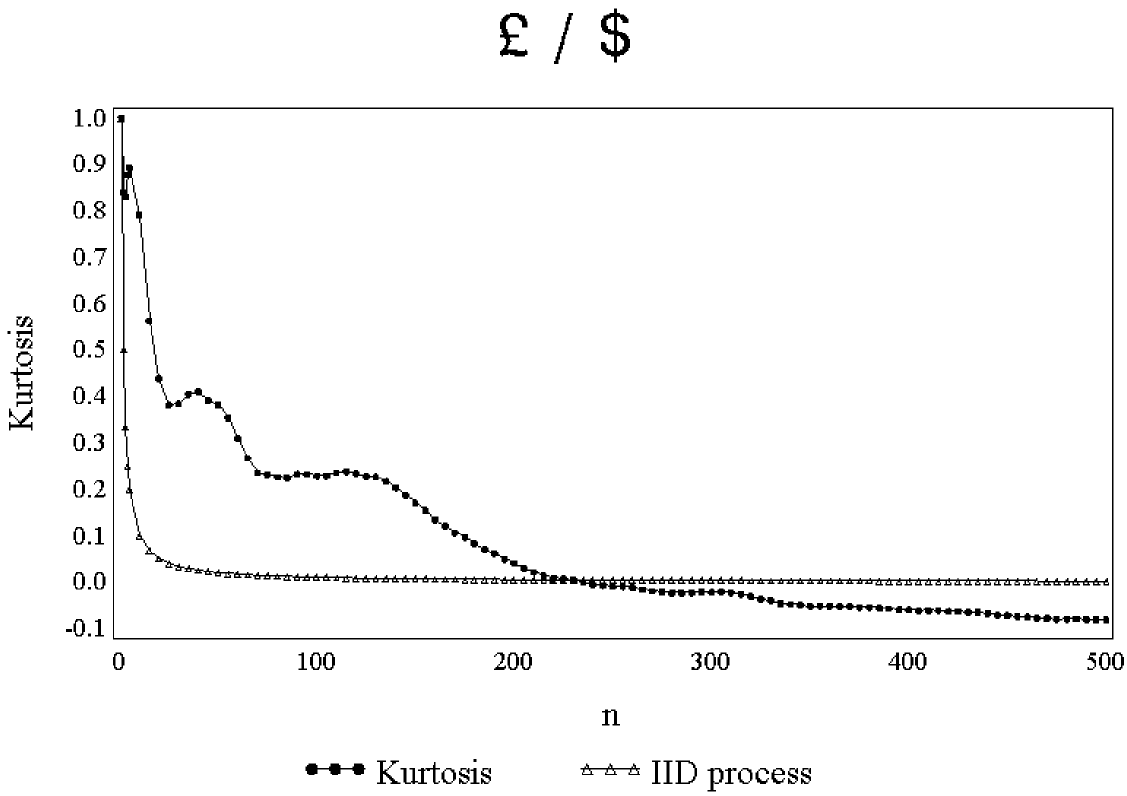

Fig. 8. Kurtosis versus *n* and the curve of an IID process

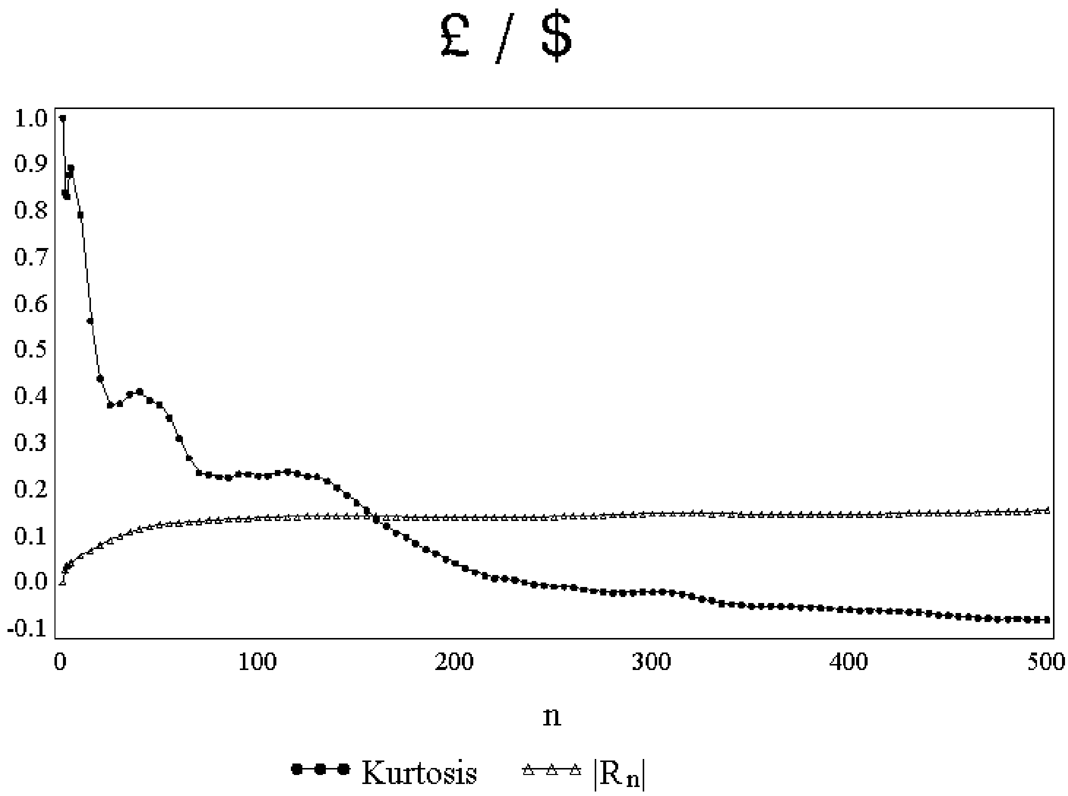

Fig. 9. Comparison between the kurtosis and term $|R_n|$

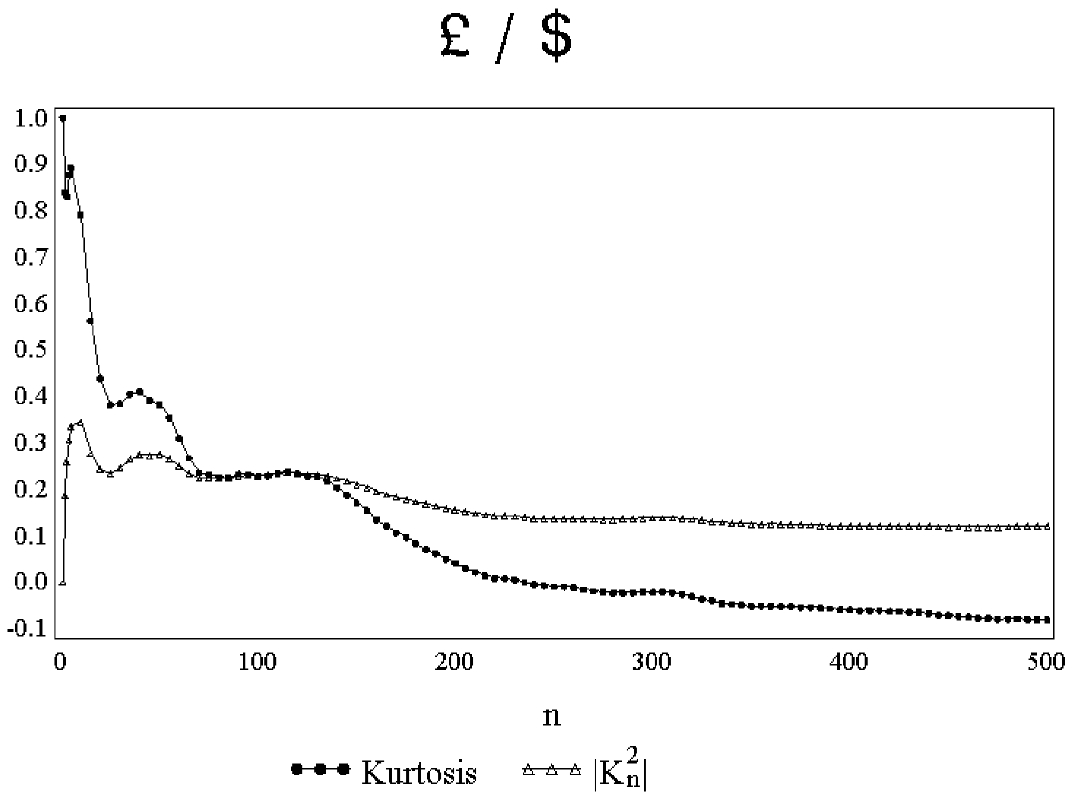

Fig. 10. Comparison between K and $|K_n^2|$

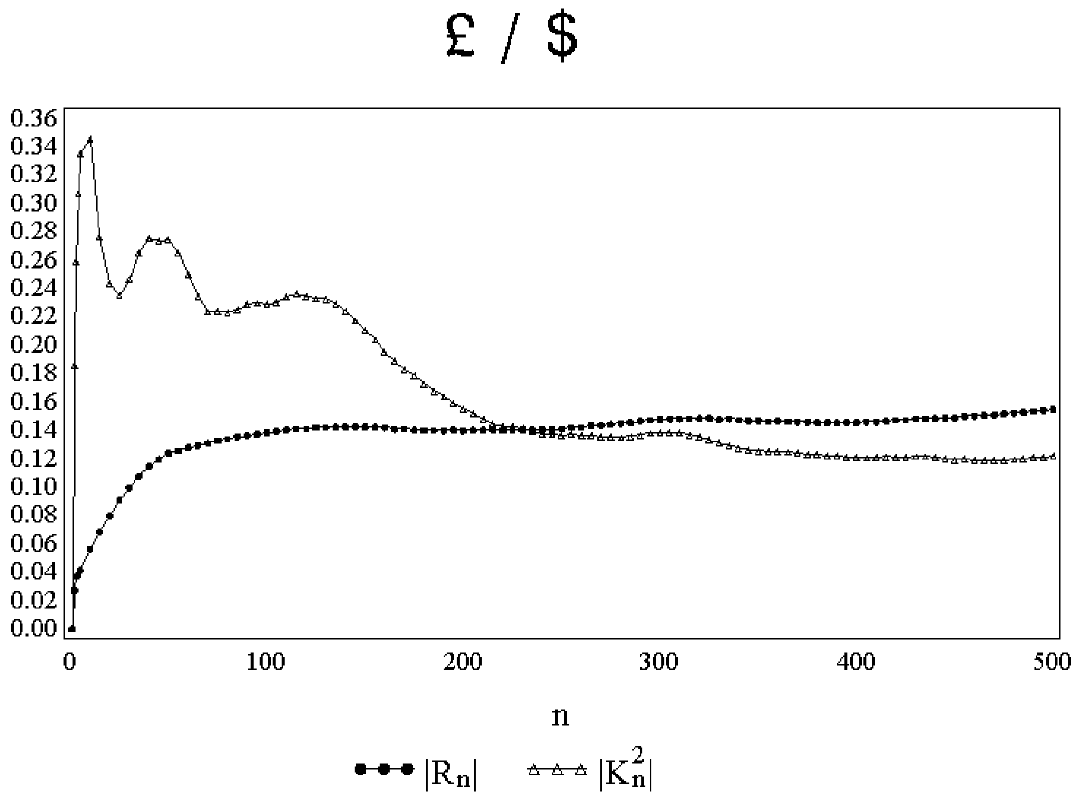

Fig. 11. Comparison between $|R_n|$ and $|K_n^2|$

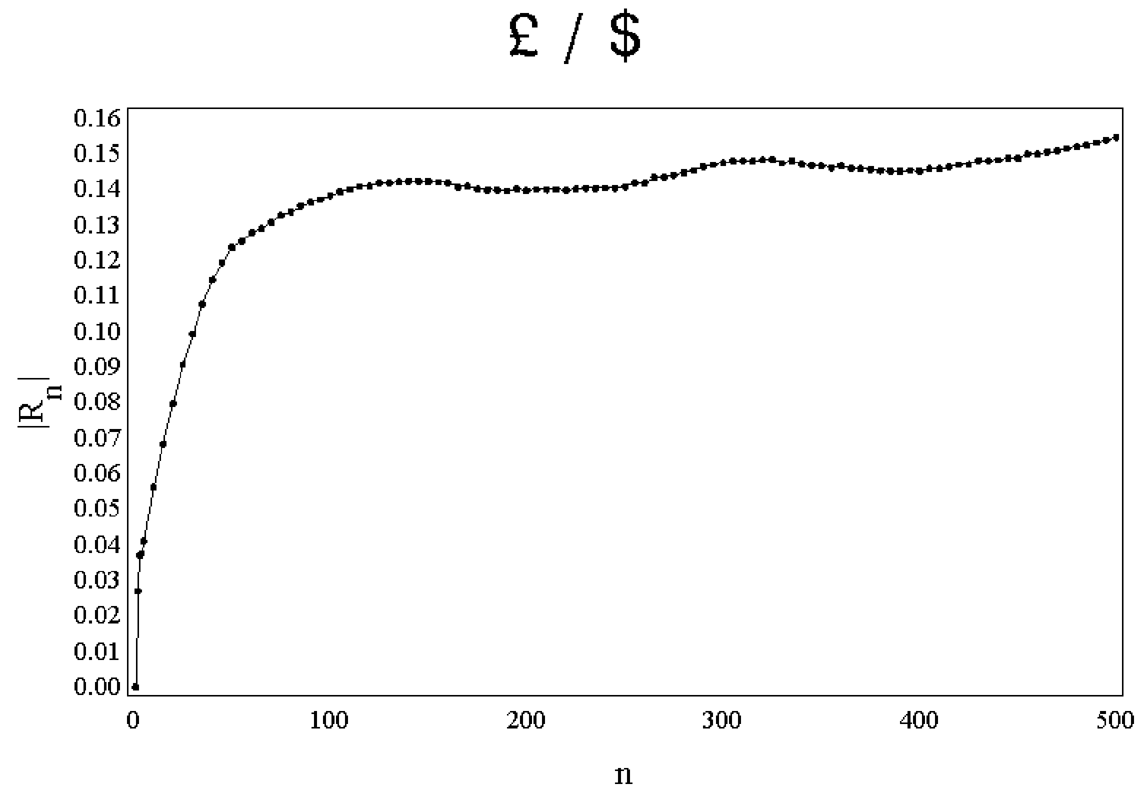

Fig. 12. $|R_n|$ versus n

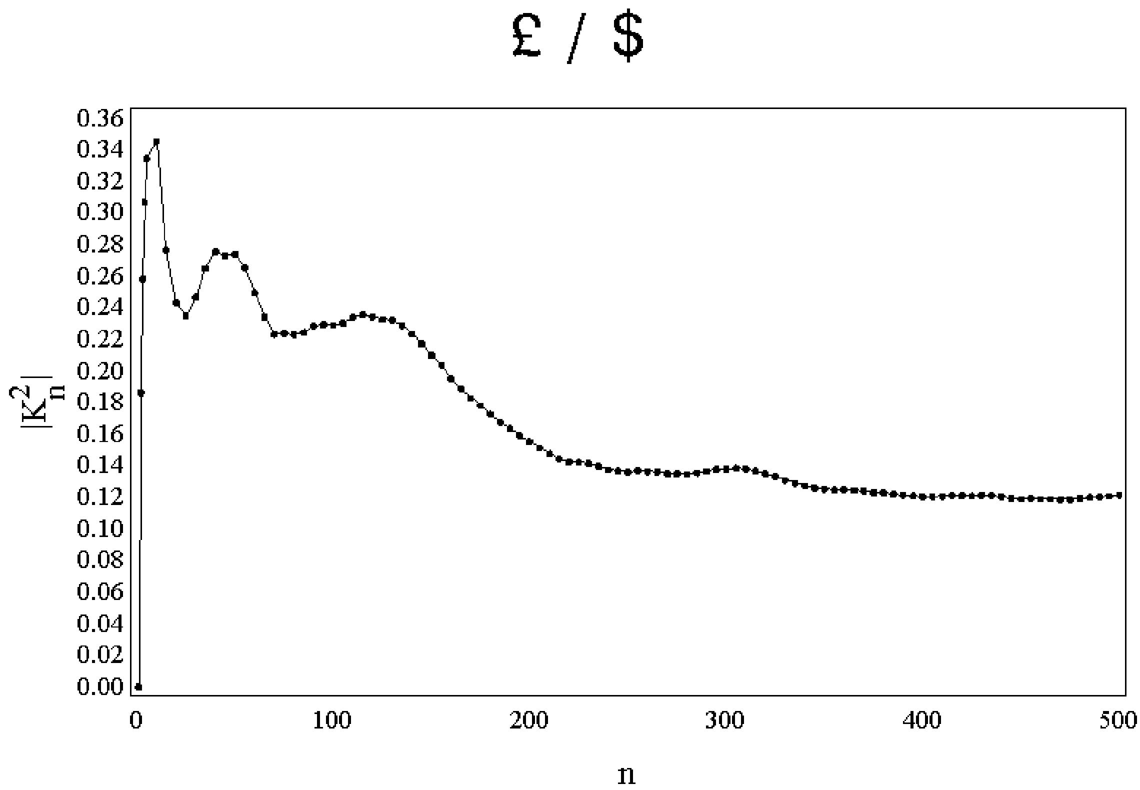

Fig. 13. $|K_n^2|$ versus n

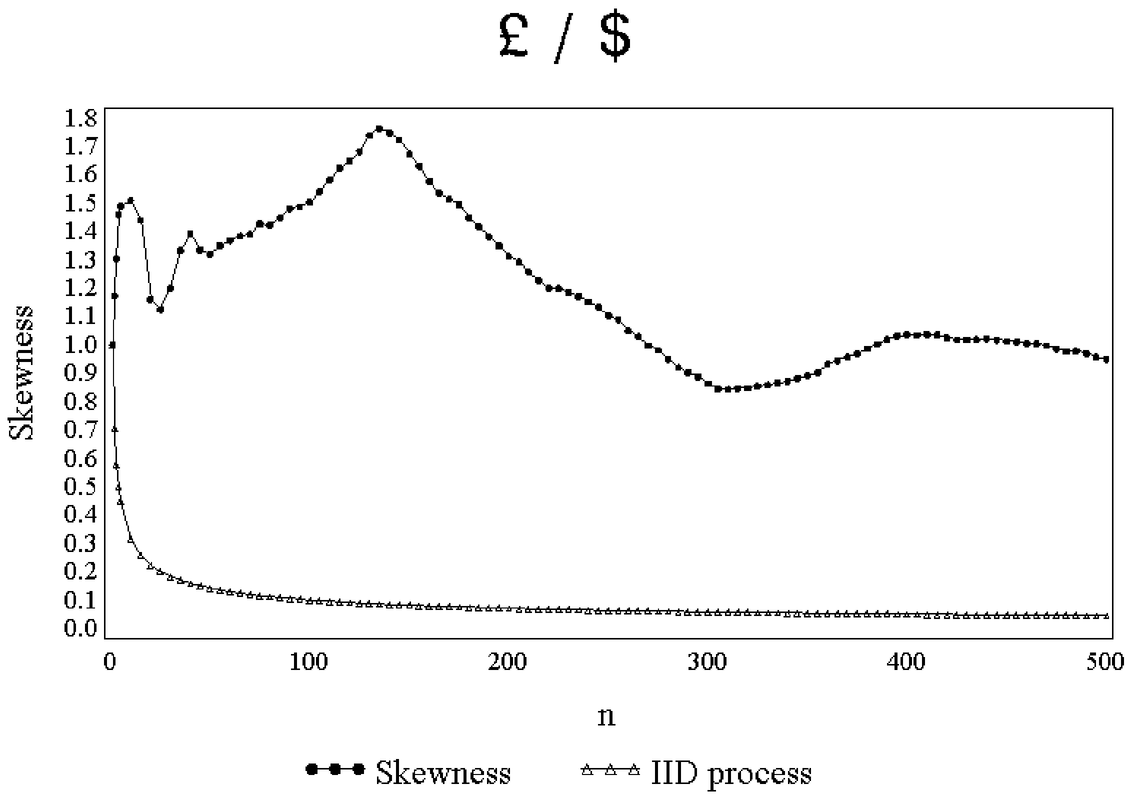

Fig. 14. Skewness versus *n* and the curve of an IID process

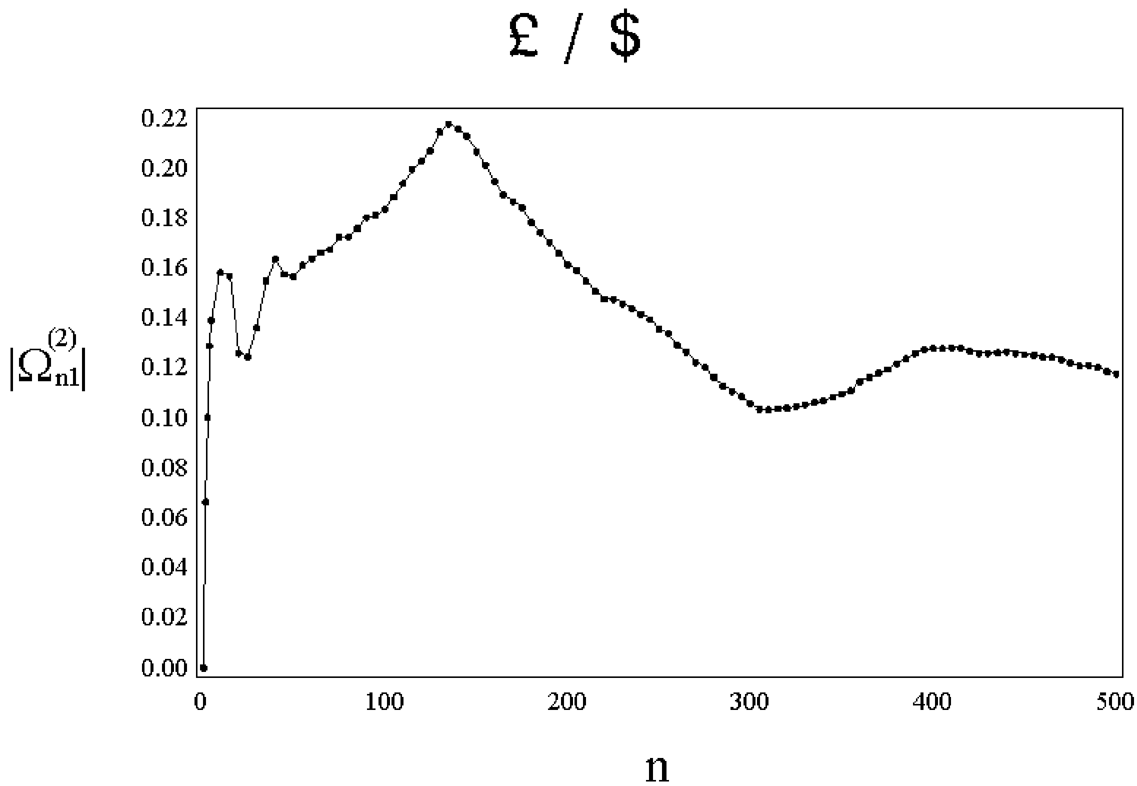

Fig. 15. $|\Omega_{n1}^{(2)}|$ versus n

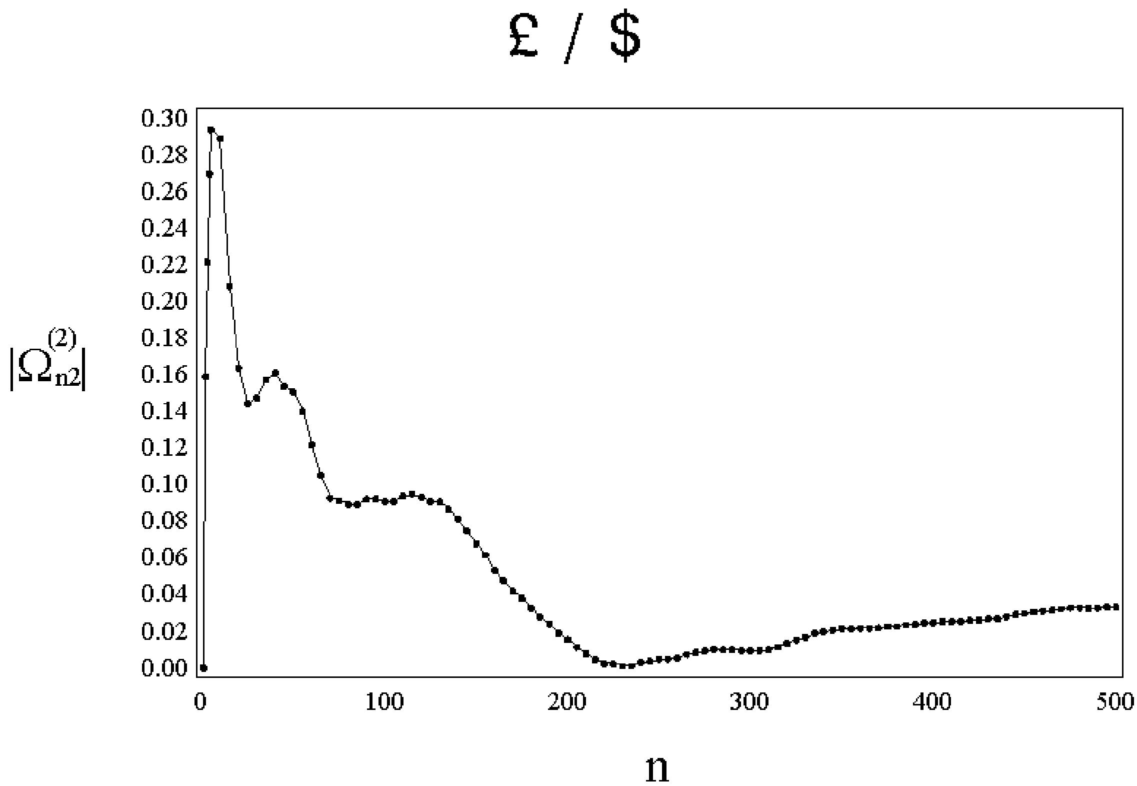

Fig. 16. $|\Omega_{n2}^{(2)}|$ versus n