The Econophysics of the Brazilian *Real*-US Dollar Rate

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Abstract

We study the statistical physics properties of the rate of exchange of the Brazilian *real* against the US dollar from both a daily and fifteen-minute perspective. We find several regularities in the form of power laws in the study of returns of the series for increasing time lags. We also evaluate the fitting of the data sets to variants of the Lévy-stable distribution. The log-periodicity hypothesis for both frequencies is assessed as well.

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1. Introduction

The statistical physics properties of actual financial prices have been assessed by an increasing number of recent papers in the realm of "econophysics" [1, 2], which is to be viewed as complementing the traditional approach of economics and finance. This paper studies the econophysics of the Brazilian *real*-US dollar rate. Throughout we take both daily and intraday rates to show that frequency does not matter for our results, a fact that is suggestive of self-similarity in the exchange rate.

Financial data in general usually exhibit leptokurtosis and thus are poorly described by the Gaussian model [3-7]. Even a biased Gaussian random walk [8] cannot account for the high peaks and fat tails of financial data. Extreme events like Black Monday are virtually impossible in a biased random walk, where the probability of more than five standard deviations is essentially zero; and Black Monday is more than 34 standard deviations [9]. Research on econophysics that take empirical data into account aims to show that catastrophic, rare events like Black Monday should be considered as part of the overall picture; they are not to be viewed as "anomalies". Even the great stock market crashes would be simply ordinary (although infrequent) events. A favorite model of financial data among econophysicists is the class of Pareto-Lévy stable distributions.

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Lévy-stable distributions were introduced by Paul Lévy in the early 1920s [10-12]. The Lévy distribution is described by four parameters, namely (1) an index of stability $\alpha \in (0,2]$, (2) a skewness parameter, (3) a scale parameter, and (4) a location parameter. Exponent α determines the rate at which the tails of the distribution decay. The Lévy collapses to a Gaussian if $\alpha = 2$. If $\alpha > 1$ the mean of the distribution exists and equals the location parameter. But if $\alpha < 2$ the variance is infinite. The *p*th moment of a Lévy-stable random variable is finite if $p < \alpha$. The scale parameter determines the width, whereas the location parameter tracks the shift of the peak of the distribution. Further details can be found elsewhere [13-15].

Although leptokurtosis could be accounted for by Lévy-stable distributions, these have never been established in mainstream finance. One reason is related to their property of infinite variance. Since volatility (standard deviation) is a central concept to finance, it is useful for the variance to be finite. (The debate in the early days of modern finance can be appreciated in Cootner [16].)

Modifications to the ordinary Lévy-stable distribution have been suggested in the econophysics literature as an attempt to overturn the drawback of infinite variance. They were pioneered by Mantegna and Stanley [17], who carried out an abrupt truncation on the distribution tails. They have applied their truncated Lévy flight (TLF) to the S&P500 index [18], and their approach has since been employed and extended to other asset prices [19-22]. Some studies have also cast doubt on the hypothesis of Lévy-stability of returns and found an α greater than two [23-32]. Yet α might be overestimated in such studies. A value found around three, for instance, may really mean an $\alpha \approx 1.8$ [13].

The TLF is a stochastic process with finite variance and characterized by scaling relations in a large but finite interval. A TLF is not a stable process once only Lévy distributions are stable. Because it has finite variance, the TLF slowly converges to a Gaussian equilibrium as implied by the classic central limit theorem. The TLF (and its characteristic function (CF)) is not infinitely divisible because the truncation of the distribution is abrupt. Yet it is still possible to set up a Lévy flight with a smooth (exponential) cutoff that produces an infinitely divisible CF [32]. This has been dubbed a smoothly truncated Lévy flight (STLF). Indeed, processes whose CF has an exponential form are infinitely divisible. The cutoff of the STLF is carried out by asymptotic approximation of a stable distribution valid for large values [33]. The cutoff can still gradually truncate a Lévy flight (GTLF) [34, 35], and this brings even fatter tails. Also, an exponentially damped Lévy flight (EDLF) [36] (suggested by some of us) deviates from the Lévy-stable distribution in both a smooth and gradual fashion.

The sharp cutoff of the TLF makes moment scaling approximate and valid for a finite time interval only; for longer time horizons, scaling breaks down. And the breakdown depends not only on time but also on moment order.

Whether scaling is single or multiple depends on how the Lévy is broken. While the abruptly truncated Lévy flight (the TLF itself) exhibits mere single scaling [33], the STLF can show multiscaling [37]. Either an abruptly TLF or an EDLF is able to fit data sets of 30 daily exchange rates against the US dollar [21, 36]. And multiscaling [38-43] with the EDLF has been found to be pervasive for the data sets above [44] together with a *false* euro [45].

It is worth studying in greater detail the sluggish convergence of the TLF to the Gaussian regime. Here we describe an approach based on analysis of the CF of the process.

And we show that the ultraslow convergence is linked to the presence of nonlinear autocorrelations. What is more, we present a straightforward technique to determine the distance of any distributions to the Gaussian one.

And what if crashes are outliers? This departures from the econophysics approach above where extreme events are thought to be tracked by a Lévy. A sanguine, recent hypothesis is for the crashes to be deterministic and governed by log-periodic formulas [46, 47]. And this log-periodicity hypothesis of crashes that are outliers also departures from conventional statistics that has a well established tradition in the study of extreme events. Here we evaluate the log-periodicity hypothesis for the *real*-dollar rate.

The rest of this paper is organized as follows. Section 2 presents data. Section 3 discusses efficiency and reckons Hurst exponents. Section 4 displays regularities in both the time of autocorrelation and an index of algorithm complexity. Section 5 reports on power laws in statistical moments. Section 6 employs the Lévy to the data. Section 7 presents our methodology in connection with the *real*-dollar data to determine the distance of the process to the Gaussian regime and discuss the role of nonlinear autocorrelations. Section 8 adjusts log-periodic formulas to the series. And Section 9 concludes.

2. Data

Data sets are for the daily and intraday frequencies. The daily series covers the period from 2 January 1995 to 31 December 2003. The set comprises 2259 data points obtained from the Federal Reserve website. The 15-minute set comprises 9327 data points from 9:30AM of 19 July 2001 to 4:30PM of 14 January 2003. Figures 1a and 1c display the raw data of the two sets of data, whereas Figures 1b and 1d show single returns ($\Delta t = 1$). The daily series features a major structural break to the naked eye, corresponding to the Brazilian currency crisis of 13 January 1999. Some argue that there exists extra endogenous breaks that can be detected by a Markov-switching regime analysis [48]. The intraday series is sampled from the more recent floating period and is thus more likely to be trendless, as discussed below.

Our major discoveries in this paper are related to the regularities found in the study of returns $Z_{\Delta t}(t) \equiv Y(t+\Delta t) - Y(t)$ for increasing Δt. Studying returns this way has been previously put forward as an alternative to the investigation of the tails in characterizing a functional form of probability density functions [18].

3. Efficiency and Hurst exponent

Financial market efficiency is meant that an exchange rate shows no trends if information is widespread. Generally, under informational efficiency an asset price embodies all available information. So-called weak efficiency occurs if the price embodies the information accruing from historic series. A semi-strong efficiency encompasses the use of historic series and the notion that the market moves according to the availability of public information. And strong efficiency occurs if the market embodies all information available, whether public or insider information.

Departures from efficiency translate into the possibility of excess returns. In an efficient market excessive profits are not possible for those with special information because the price would adjust at the same pace as the arrival of new information.

Informational efficiency is usually associated with (types 1, 2, and 3) random walks. A random walk 1 (RW1) assumes increments that are independent and identically distributed (IID). Since the increments belong to the same distribution, many argue that the RW1 is not appropriate for series with structural breaks. And that is precisely the case of the daily *real*-dollar rate. As observed, it experienced (Figure 1a) at least a major break. Unless the distribution is a stable Lévy, one might argue that an identical distribution for the entire time period is unlikely for such series. Thus an RW2 would be appropriate once it assumes independent increments coming from distinct distributions. But independence for both RW1 and RW2 is meant more than just assuming that increments are uncorrelated.

So an RW3 would be even better once it assumes distinct distributions and increments that are just uncorrelated. Yet despite the fact that return levels are here uncorrelated, squared increments still present some type of autocorrelation.

Previous work has found evidence for weak efficiency and for an RW3 for the Brazilian daily foreign exchange market [48, 49]. As will be seen, such a result matches our own findings regarding the Hurst exponents of the series.

The Hurst exponent [50-52] is related to how the value of a stochastic variable moves away from its initial position. A Hurst exponent of 0.5 gives an indication of a random walk even if a series is not normally distributed, whereas a value of between 0.5 and 1.0 suggests a persistent, trend-reinforcing series, and a value less than 0.5 indicates antipersistence, i.e. past trends tend to reverse in the future. Generally values of the Hurst exponent departing from 0.5 are meant that data points are not independent, regardless of Gaussianity. Here a current data point carries out memory, i.e., information from the precedent points. And this is not short range memory, but long range memory that can theoretically last forever.

Long range memory distorts optimal consumption and portfolio decisions, which become highly sensitive to investment horizon. It also renders the pricing of financial derivatives (through methods such as the Black-Scholes model) useless. Also, tests based on the capital asset pricing model and arbitrage pricing theory get meaningless in the presence of long range memory.

It is worth emphasizing that, as seen, an RW3 weak efficiency is compatible with the notion that a series is autocorrelated. For the simple returns ($\Delta t = 1$) of the daily *real*-dollar rate, we have reckoned, for entire samples, a Hurst exponent $H = 0.5561864$; for the intraday data, $H = 0.5176148$.

Note that the values of the Hurst exponent are compatible with the finding of weak efficiency in the Brazilian *real*-dollar market. For the intraday data, the Hurst is even closer to 0.5. However the exponents are not exactly at 0.5 and thus there is room for autocorrelations and other sort of determinism in the series.

As Δt is raised in the definition of returns, there is heightened aggregation, and the Hurst exponents are expected to grow as a result. Surprisingly, there are power laws governing the growth pace of the exponents. And the laws hold regardless of structural breaks in the (daily) series. Figures 2a and 2b display this finding in log-log plots of the daily and intraday series respectively.

The Hurst exponents above have been calculated using *Chaos Data Analyzer* [53], whose program does not rely on rescaled range (R/S) analysis. (The technique employed is described elsewhere [54].) They can be alternatively reckoned using R/S analysis (Figures 3a and 3b). For the daily data, $H = 0.586396$, and for the intraday data, $H = 0.622155$.

These figures are greater than the ones presented above, and suggestive of both extra autocorrelation in data and less efficiency.

The R/S analysis has been criticized for not properly distinguishing between short range and long range memory [55]. Suggested modifications [55], however, present a bias against the hypothesis of long range dependence [56, 57]. A recent alternative is for the Hurst exponents calculated by R/S analysis to be filtered by an AR(1)–GARCH(1,1) process [58].

Hurst exponents could be reckoned over time to evaluate whether the series are getting more or less efficient [58]. Here histograms will accompany the plots of the Hurst exponents to see whether these are normally distributed, in which case variations in the exponent should be ascribed to measurement errors.

Figure 3c shows the evolution of the Hurst over time for the daily *real*-dollar returns. As in Ref. [58], a four year (1008 observations) time window has been chosen. There are two patterns in Figure 3c. Prior to observation 1010 the Hurst unambiguously approaches 0.5, which is meant that the market gets more efficient. But from that point on the Hurst gets away from 0.5 and so the market becomes less efficient. This finding makes sense.

Indeed Figure 1a shows that the raw data until December 1998 (observation 1010) follow a deterministic rule in which the Brazilian central bank devalues the *real* in 0.003 per cent on a daily basis. Yet Figure 1b shows that data on returns are stochastic. But since market participants could easily use the information of a predictable daily devaluation, market efficiency is more likely. Moreover as the *Real* Plan of 1st July 1994 becomes increasingly successful, credibility of the central bank is heightened, and this explain why the foreign exchange market gets more efficient over time.

After the currency crisis of 13 January 1999, the *real*-dollar rate is let to float. Several shocks, ranging from domestic macroeconomic and political problems to contagion of foreign currency crises, have made the processing of new information hitting the market more difficult. And this explains why the foreign exchange market is becoming less efficient since then. The histogram in Figure 3c shows that such an analysis is robust.

Calculation is repeated for the intraday data (Figure 3d). Here a time window of 6085 data points (nearly one year) has been used. Unfortunately errors of measurement are interfering (bottom of Figure 3d) and the Hurst exponent behavior shown in top of Figure 3d gets meaningless.

For the daily data, analysis is filtered by an AR(1)–GARCH(1,1) process given by $Z_1(t) = a + \psi Z_1(t-1) + \varepsilon(t)$, $\varepsilon(t) = e(t)\sqrt{h(t)}$, $h(t) = b + \Theta_1 \varepsilon^2(t-1) + \Theta_2 h(t-1) + \Psi D(t)$, where a, b, ψ, Ψ, Θ_1, Θ_2 are parameters to be estimated, $h(t)$ is the conditional variance of residuals, and $e(t)$ is assumed to be normal and independent of $e(t')$, for $t \neq t'$. In particular, $D(t)$ is set to 0 if $t \leq 1010$ (to capture the first regime), and $D(t)=1$ otherwise. For the intraday data, $\Psi = 0$.

4. Autocorrelation time and complexity

Because the Hurst exponents calculated are compatible with the presence of autocorrelation, we investigate the behavior of the autocorrelation time. The latter measures how much current observations are dependent on previous ones. The autocorrelation time is expected to increase with Δt. But what is also surprising is that such a growth rate is governed by power laws. Figure 4a shows the power law for the daily series, and Figure 4b displays the law for the intraday data.

A concept related to both the Hurst exponent and autocorrelation time is that of an *LZ* (Lempel-Ziv) complexity relative to Gaussian white noise [59, 60]. An *LZ* index of zero is associated with perfect predictability, and an index value of about one gives piece of evidence of high complexity (genuine randomness). To reckon the algorithmic complexity of a series, every data point is converted into a binary figure and then compared to the median of the entire series.

For the single returns ($\Delta t = 1$) of the daily *real*-dollar rate we have reckoned an *LZ* index $LZ = 1.04107$; for the intraday data, $LZ = 0.9999905$. Such figures are consistent with both RW3-type weak efficiency and the values for the Hurst exponents above. As Δt is raised in the definition of returns, heightened aggregation introduces structure in the series, these get more predictable, and thus the *LZ* index decays to zero. What is more (and that is surprising), power laws govern the decays (Figures 5a and 5b).

5. Power laws in statistical moments

Not surprisingly, both the mean and standard deviation (volatility) grow as Δt is raised. What is remarkable is that these changes are governed by power laws. Figures 6 and 7 show these findings for both the daily and intraday data. Accordingly such statistical moments can be expressed as $\omega(\Delta t)^{\beta}$, where the effect of ω on the moments is larger the greater Δt is [19]. Note that these scale-free power laws are consistent with non-Gaussian scaling (slope $\neq -0.5$).

In the benchmark study of Mantegna and Stanley [18], means are assumed to be fixed at zero for increasing Δt . They then take the "probability of return to the origin" $P(Z = 0)$, which is a method for rescaling the Lévy to reveal its self-similarity. We have warned [19] that, since the peak of a distribution is not exactly located at $Z = 0$, one should take $P(Z = \omega(\Delta t)^{\beta})$ to replace the probability of return to the origin. Neglecting this fact does not change Mantegna and Stanley's results a great deal however, thanks precisely to the existence of a power law in the means and also to the fact that ω and β are tiny. Thus we can experimentally find $P(0)$ for the daily *real*-dollar rate by using a small threshold value v, which is defined such that $P(0) \approx P(| Z | \leq v)$ [19]. Figures 8a and 8b show the power law in the probability of return to the origin (as redefined above) for both data sets.

6. The hypothesis of a Lévy distribution

The non-Gaussian scaling of the power laws in the probability of return to the origin is at first consistent with the presence of a truncated Lévy flight in our sets of data. Using the TLF as in Ref. [1], i.e. $P(Z) \equiv L_{\alpha}(Z, \Delta t) \equiv (1/\pi)\int_0^{\infty} \exp(-\gamma \Delta t q^{\alpha}) \cos(qZ) dq$, Figure 9a displays the logarithm of the probability density function (PDF) of daily returns. (And Figure 9b for intraday returns.) Increases in time horizon range from Δt = 1, 2, and 5 trading days (a week) to 240 trading days (a year). A spreading of the PDFs characteristic of any random walk is observed. We have calculated $\alpha = 0.89059$ and $\gamma = 0.003707604$. Parameter $\alpha < 2$ is compatible with a Lévy for the modal region of the distribution.

Thanks to the scaling in the probability of return to the origin, the PDFs in Figure 9a can be plotted in scaled units $P(Z)$. Given the scaling index α, the data are made to collapse onto the $\Delta t = 1$ distribution (Figure 10a). Thus the Lévy PDF is seen to model the central region of the distribution within a finite time interval. (Figure 10b presents the PDFs for the intraday data.)

The EDLF adjusts to the same data set. But here estimation of parameters α and γ departures from the estimation approach of plotting the probability of return to the origin against Δt. Now parameters α and γ are estimated by maximum likelihood, and the other parameters by nonlinear least squares. We get $\hat{\alpha} = 0.5960$ and $\hat{\gamma} = 0.00157227$ for the daily series. Figure 11a displays the log of differences showing how the observed log PDFs of the daily returns deviate from the original log Lévy process. The continuous lines are the fittings using the variance and $Z_s = \Delta t^{-1/\hat{\alpha}} Z_{\Delta t}$. (Figure 11b is for the intraday data.)

Figure 12a shows that the EDLF fits the daily *real*-dollar rate data reasonable well. Note that the larger dispersion at the tails area in Figure 12a is partly due to the equal histogram bins taken in the scale of Z. The dispersion could be significantly reduced if we had taken equal bins in $\ln Z$ [14]. (Figure 12b is the EDLF for intraday returns.)

Assuming that $\ln[\varphi(r)] \approx -\gamma\Delta t|r|^{\alpha}$ for $0 < r < \alpha$, the "estimated" norm in logs of the characteristic function is $\ln||\varphi(r)||$, and then we can expect that $\ln|| \hat{\varphi}(r) || = -\gamma\Delta t|r|^{\alpha}$. Figure 13a displays sample ratios $R(r,\Delta t) = <|Z_{\Delta t}|^{r}>/<|Z_1|^{r}>$. Ratios $R(r, \Delta t)$ are shown for several values of r in a log-log plot of the daily *real*-dollar rate. Values range from $r = 0.0$ to $r = 3.0$ at intervals of 0.2; the bottom line corresponds to $r = 0.0$, and the top one to $r = 3.0$. (Figure 13b is for the intraday data.)

Fitting $\ln R(r, \Delta t) = \xi \ln\Delta t$ for every r gets the corresponding scaling exponents. Figure 14a displays the estimated multiscaling exponent ξ for the daily data. An approximate linear behavior for all r presents evidence of mere single scaling. A linear behavior for initial values of $r < \alpha$, followed by a nonlinear pattern after $r > \alpha$, tracks the presence of multiscaling. As can be seen, the data exhibit multiscaling. (Figure 14b is for the intraday returns.)

Figure 15a shows estimated ratios $\ln||\varphi(r)||$ of the daily *real*-dollar returns for r = 0.0–3.0 at intervals of 0.2 (for each plot, the upper line corresponds to $r = 0.0$, and the bottom one to $r = 3.0$). By fitting $\ln|| \hat{\varphi}(r) || = \zeta\Delta t$ for every r, the estimated values of ζ versus $\kappa(\alpha) = |r|^{\hat{\alpha}}$ are plotted in Figure 16a. The daily *real*-dollar returns present linear dependence for all $\kappa(\alpha) < 3$, and fail to be followed by a nonlinear pattern. So as for ζ, the data do not feature multiscaling. (Figures 15b and 16b are for the intraday data.)

7. Convergence to the Gaussian regime and the role of autocorrelations

Taking our series into account, now we assess the problem of how distant a process currently is from the Gaussian regime as well as the role of autocorrelations in the convergence speed of the process.

First we take returns $Z_{\Delta t}(t) \equiv Y(t + \Delta t) - Y(t)$. Note that $\Delta t = 1$ can (for instance) mean *one day*, thereby Z_5 is the sum of five daily variations, and so on. More generally, $Z_{\Delta t}$ is thought of as the sum of Δt random variables x_t, i.e. $Z_{\Delta t} = \sum_{t=1}^{\Delta t} x_t$, where x_t has zero mean. For "reduced" variables $\bar{x} = x/\mu_2^{1/2}$ (where μ_2 is the variance of x), Lévy [10, 11] shows that the CF, $\varphi(r)$, of a process with finite second moment can be written as $\varphi(r) = e^{-r^2(1+W(r))/2}$, where $W(0)=0$. For the CF of x_t we can thus write $\varphi_t(r) = e^{-\mu_{t,2}r^2(1+W_t(\mu_{t,2}^{1/2}r))/2}$. And for $Z_{\Delta t}$ the CF is $\Phi_{\Delta t}(r) = e^{-\nu_{\Delta t,2}r^2(1+\Omega_{\Delta t}(\nu_{\Delta t,2}^{1/2}r))/2}$, where $\nu_{\Delta t,2}$ is the variance of $Z_{\Delta t}$. We denote the statistical moments of order p of x_t and $Z_{\Delta t}$ as $\mu_{t,p} = \langle x_t^p \rangle$ and $\nu_{\Delta t,p} = \langle Z_{\Delta t}^p \rangle$ respectively. We consider too that $\eta_{\Delta t,p} = \sum_t \mu_{t,p}$.

For independent variables it holds true that $\Phi_{\Delta t}(r) = \varphi_1(r)\cdots\varphi_{\Delta t}(r)$. But this does not hold for autocorrelated processes, in which case the CF must have an additional term such as

$$\Phi_{\Delta t}(r) = C_{\Delta t}(r)\varphi_1(r)\cdots\varphi_{\Delta t}(r) \tag{1}$$

with $C_{\Delta t}(r) = 1$ for an independent process.

We can expand the CF of x_t in series to obtain

$$\varphi_t(r) = 1 + \frac{I^2}{2!}\mu_{t,2}r^2 + \frac{I^3}{3!}\mu_{t,3}r^3 + \cdots \tag{2}$$

We also assume that

$$C_{\Delta t}(r) = e^{-r^2\left(-2C_{\Delta t,2}+W_{\Delta t}(r)\right)/2} = 1 + C_{\Delta t,2}r^2 + C_{\Delta t,2}r^3 + \cdots \tag{3}$$

We can do the same for the CF of $Z_{\Delta t}$. Expanding $\Phi_{\Delta t}(r)$ gives

$$\Phi_{\Delta t}(r) = 1 + \frac{I^2}{2!}\nu_{\Delta t,2}r^2 + \frac{I^3}{3!}\nu_{\Delta t,3}r^3 + \cdots \tag{4}$$

Plugging Eqs. (2)-(4) in Eq. (1), and comparing equal order terms one gets

$$C_{\Delta t,2} = -\frac{1}{2}\left(\nu_{\Delta t,2} - \eta_{\Delta t,2}\right),\quad C_{\Delta t,3} = -\frac{I}{2}\left(\nu_{\Delta t,3} - \eta_{\Delta t,3}\right)$$

$$C_{\Delta t,4} = \frac{1}{4!}\left(\nu_{\Delta t,4} - \eta_{\Delta t,4}\right) - \frac{1}{2!2!}\left(\eta_{\Delta t,2}\left(\nu_{\Delta t,2} - \eta_{\Delta t,2}\right) + \gamma_{\Delta t}\right) \tag{5}$$

where $\gamma_{\Delta t} = \sum_{i=1}^{\Delta t-1}\sum_{j=i+1}^{\Delta t,}\mu_{i2}\mu_{j2}$. If we write $W_{\Delta t}(r) = IW_{\Delta t,1}r + W_{\Delta t,2}r^2 + O(r^3)$, from Eqs. (3) and (5) we get

$$W_{\Delta t,1} = \frac{1}{3}\left(\nu_{\Delta t,3} - \eta_{\Delta t,3}\right),\quad W_{\Delta t,2} = \frac{1}{4}\left(\nu_{\Delta t,2} - \eta_{\Delta t,2}\right)^2 - 2C_{\Delta t,4} \tag{6}$$

Plugging Eqs. (5) and (6) back in the CF of $Z_{\Delta t}$ yields

$$\Phi_{\Delta t}(r) = e^{-r^2\left(\nu_{\Delta t,2} + \sum_{t=1}^{\Delta t,}\mu_{t,2}W_t(\mu_{t,2}^{1/2}r) + W_{\Delta t}(r)\right)/2} \tag{7}$$

After writing the CF of the reduced variable as $\overline{\Phi}_{\Delta t}(r) = e^{-r^2\left(1+\Omega_{\Delta t}^{(1)}(r)+\Omega_{\Delta t}^{(2)}(r)\right)/2}$ and reminding that $\overline{\Phi}_{\Delta t}(r) = \Phi_{\Delta t}\left(\frac{r}{\nu_{\Delta t,2}^{1/2}}\right)$ one has

$$\Omega_{\Delta t}^{(1)}(r) = \frac{1}{\nu_{\Delta t,2}} \sum_{i=1}^{\Delta t} \mu_{i,2} W_i \left(\left(\frac{\mu_{i,2}}{\nu_{\Delta t,2}} \right)^{1/2} r \right),$$

$$\Omega_{\Delta t}^{(2)}(r) = \frac{1}{\nu_{\Delta t,2}} W_{\Delta t} \left(\frac{r}{\nu_{\Delta t,2}^{1/2}} \right) \tag{8}$$

Function $\Omega_{\Delta t}^{(1)}(r)$ matches that for uncorrelated series, i.e., as $\Delta t \to \infty$ it approaches $W(0) = 0$ in accordance with the central limit theorem (CLT). Term $\Omega_{\Delta t}^{(2)}(r)$ is related to the autocorrelations. It gives precisely the CF of the sum variable, which in turn can be used to obtain the PDF as $\Delta t \to \infty$.

Now we relate $\Omega_{\Delta t}^{(2)}(r)$ to *nonlinear* autocorrelations, which can be captured by

$$\langle p_1 p_2 ... p_m \rangle_{\Delta t} = \sum_{i_1 ... i_m}^{\Delta t} \left(\langle x_{i_1}^{p_1} ... x_{i_m}^{p_m} \rangle - \langle x_{i_1}^{p_1} \rangle ... \langle x_{i_m}^{p_m} \rangle \right) \tag{9}$$

where $p_1 p_2 \cdots p_m$ are positive integers, and $i_1 \neq i_2 \neq \cdots \neq i_m$. After writing $\Omega_{\Delta t}^{(2)} = \Omega_{\Delta t,1}^{(2)} rI + \Omega_{\Delta t,2}^{(2)} r^2$ it can be shown that

$$\Omega_{\Delta t,1}^{(2)} = \frac{1}{3} \frac{\nu_{\Delta t,3} - \eta_{\Delta t,3}}{\nu_{\Delta t,2}^{3/2}} = \frac{1}{3} \frac{\langle 111 \rangle_{\Delta t} + 3\langle 12 \rangle_{\Delta t}}{\nu_{\Delta t,2}^{3/2}}$$

$$\Omega_{\Delta t,2}^{(2)} = \frac{1}{4} \left(1 - \frac{\eta_{\Delta t,2}^2}{\nu_{\Delta t,2}^2} \right) - \frac{1}{12} \frac{\nu_{\Delta t,4} - \eta_{\Delta t,4} - 6\gamma_{\Delta t}}{\nu_{\Delta t,2}^2}$$

$$= \frac{1}{4} \left(1 - \frac{\eta_{\Delta t,2}^2}{\nu_{\Delta t,2}^2} \right) - \frac{1}{12} \frac{\langle 1111 \rangle_{\Delta t} + 6\langle 112 \rangle_{\Delta t} + 4\langle 13 \rangle_{\Delta t} + 3\langle 22 \rangle_{\Delta t}}{\nu_{\Delta t,2}^2} \tag{10}$$

where $\Omega_{\Delta t,1}^{(2)}$ and $\Omega_{\Delta t,2}^{(2)}$ are functions of third- and fourth-order autocorrelations respectively.

Due to the presence of the nonlinear autocorrelations, either $\Omega_{\Delta t,1}^{(2)}$ or $\Omega_{\Delta t,2}^{(2)}$ may remain bounded above zero as Δt gets larger. From these results it turns out that the limit distribution may not be a Gaussian. Furthermore, the norm of $W(r)$ in the expression for the CF gives a good measure of the distance of a PDF to the Gaussian, where $W(r) = 0$. For a given δ, the distance between a distribution f and the Gaussian can be estimated by

$$D(f, Gauss) = \int_{-\delta}^{\delta} \sqrt{W_R(r)^2 + W_I(r)^2}\, dr \tag{11}$$

Expression $W(r) = W_R(r) + IW_I(r)$ can be further expanded in series to give [21] $W_R(r) = (-r^2/12)K + O(r^4)$ (where $K \equiv \left(\mu_4 / \mu_2^2\right) - 3$ is the kurtosis) and $W_I(r) = (r/3)Sk + O(r^3)$ (where $Sk \equiv \mu_3 / \mu_2^{3/2}$ is the skewness). Thus the leading terms in $\Omega_{\Delta t}$ are the kurtosis and skewness of the sum variable $Z_{\Delta t}$. After remembering that such quantities are zero for a Gaussian, our results mean that the distance to the Gaussian is given by how distant K and Sk are from zero, which sounds quite appropriate.

Figure 17a shows the curve of Eq. (11) with $\delta = 1$ for the daily variations of the *real*-dollar rate. And Figure 17b presents the same curve for the intraday data. For the daily variations the function is somewhat constrained to some real value which prevents termalization ($W(0) = 0$) to take place. Yet the intraday data seem to converge much faster to the Gaussian regime.

Figures 18a and 18b present the kurtosis and Figures 19a and 19b show the skewness. These are the leading terms in the expansion of $W(r)$. The curve of an IID process is shown for comparison. The intraday data set is closer to an IID, in agreement with the results in Figures 17. Note that the skewness is bounded to some real value in both cases although it is of an order of magnitude smaller for the intraday data.

Figures 20a and 20b present $\Omega^{(2)}_{\Delta t,1}$. And Figures 21a and 21b display $\Omega^{(2)}_{\Delta t,2}$. Both terms are larger for the daily variations, which is also consistent with the results in Figures 17.

8. Log-periodicity

What if extreme events are not in the Lévy tails, and are outliers? Sornette and colleagues [53, 54] put forward the sanguine hypothesis that crashes are deterministic and governed by log-periodic formulas.

Their one-harmonic log-periodic function is $\ln Z(\tau) = A + B\tau^{\lambda} + C\tau^{\lambda}\cos[\theta\ln(\tau) + \phi_1]$, where $\tau = t - t_c$. And the two-harmonic log-periodic function is given by $\ln Z(\tau) = A + B\tau^{\lambda} + C\tau^{\lambda}\cos[\theta\ln(\tau) + \phi_1] + D\tau^{\lambda}\cos[2\theta\ln(\tau) + \phi_2]$. Here we try out, too, a three-harmonic log-periodic formula, i.e. $\ln Z(\tau) = A + B\tau^{\lambda} + C\tau^{\lambda}\cos[\theta\ln(\tau) + \phi_1] + D\tau^{\lambda}\cos[2\theta\ln(\tau) + \phi_2] + E\tau\lambda\cos[3\theta\ln(\tau) + \phi_3]$. The parameter values were estimated by nonlinear least squares.

Term $A + B\tau^{\lambda}$ is the trend across time in the equations above. And the log-periodic cycles are described by a sum of log-periodic harmonics (*LP*), i.e. $LP(\tau) = \sum_{j=1}^{J} C_j\tau^{\lambda_j}\cos[j\theta_j\ln(\tau) + \phi_j]$. Here we consider $\lambda_1 = ... = \lambda_J$ and $\theta_1 = ... = \theta_J$.

Figure 22a displays the log of the daily *real*-dollar rate from 28 August 2000 to 26 September 2003 (continuous line) together with its one-harmonic log-periodic fit (short-dashed line), two-harmonic log-periodic fit (short-dashed line), and three-harmonic log-periodic fit (long-dashed line). The three-harmonic log-periodic formula adjusts better. Yet this adjustment fails if we consider the entire series.

Figure 22b shows the fit for the intraday data using one harmonic, two harmonics, and our suggested three harmonics. As can be seen, the three-harmonic log-periodic formula adjusts better to the data. Here we have considered the starting time at 1:00PM of 31 May 2002. Parameter values for the fittings are presented in Tables 1 and 2.

9. Conclusion

This paper is a study of the econophysics of the *real*-dollar rate in both the daily and intraday frequencies. We generally get similar results for both series throughout, a fact which is consistent with the hypothesis of self-similarity in such an exchange rate.

Hurst exponents calculated for the single returns are consistent with the weak efficiency hypothesis of RW3 type. We further show the existence of power laws in the Hurst exponents as time lag is raised in the definition of returns. Studying how the Hurst exponent evolves over time allows one to uncover that, from the onset of the *Real* Plan to the currency crisis of 13 January 1999, the Hurst unambiguously approaches 0.5, which is meant that the (daily) foreign exchange market gets more efficient. But from that point on the Hurst gets away from 0.5 and so the market becomes less efficient.

Power laws are also present in the autocorrelation time and *LZ* complexity index. The means and standard deviations in the returns for heightened time horizons are governed by power laws too. These regularities are all consistent with departures from Gaussianity.

We then evaluate the hypothesis of a Lévy distribution to model both sets of data. We show that either a truncated Lévy flight or an exponentially damped Lévy flight could model the data sets. Furthermore, scaling is present, and multiscaling is likely.

We also examine the role of statistical autocorrelations in the convergence to the Gaussian equilibrium by focusing on the characteristic function of the *real*-dollar returns. The slow convergence is explained in terms of both nonlinear autocorrelations and the behavior of the kurtosis and skewness.

The log-periodicity hypothesis for both frequencies is assessed as well. We show that a three-harmonic log-periodic formula could model the data. So whether crashes are outliers is an open question for the *real*-dollar rate.

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Table 1a

Parameter	Estimate	Standard Error	95% Confidence Limits	
A	0.5991	0.00840	0.5827	0.6156
B	0.00241	0.000460	0.00151	0.00331
C	-0.00079	0.000138	-0.00106	-0.00052
θ	-8.8204	0.0573	-8.9329	-8.7079
ϕ_1	65.3479	0.3551	64.6509	66.0449
λ	0.8430	0.0280	0.7881	0.8979

Table 1b

Parameter	Estimate	Standard Error	95% Confidence Limits	
A	0.5942	0.00825	0.5780	0.6104
B	0.00298	0.000527	0.00195	0.00402
C	0.000977	0.000156	0.000670	0.00128
D	0.000170	0.000034	0.000103	0.000237
θ	-8.5215	0.0507	-8.6211	-8.4218
ϕ_1	60.3717	0.3139	59.7554	60.9879
ϕ_2	1.5558	0.6485	0.2827	2.8290
λ	0.8091	0.0257	0.7587	0.8595

Table 1c

Parameter	Estimate	Standard Error	95% Confidence Limits	
A	0.5923	0.00819	0.5762	0.6084
B	0.00325	0.000555	0.00216	0.00434
C	0.00107	0.000165	0.000744	0.00139
D	0.000187	0.000037	0.000114	0.000260
E	0.000109	0.000026	0.000059	0.000159
θ	-8.3940	0.0466	-8.4854	-8.3026
ϕ_1	59.5841	0.2880	59.0187	60.1495
ϕ_2	-0.2023	0.5954	-1.3710	0.9665
ϕ_3	-50.9140	0.8964	-52.6737	-49.1544
λ	0.7955	0.0248	0.7468	0.8441

Log-periodicity in the daily *real*-dollar rate.
Results for the one- (Table 1a), two- (Table 1b), and three- (Table 1c) harmonic log-periodic model.

Table 2a

Parameter	Estimate	Standard Error	95% Confidence Limits	
A	0.9451	0.00449	0.9363	0.9539
B	0.000609	0.000147	0.000322	0.000897
C	0.000175	0.000040	0.000096	0.000255
θ	4.8615	0.0629	4.7382	4.9849
ϕ_1	44.8125	0.4643	43.9021	45.7228
λ	0.7952	0.0299	0.7366	0.8538

Table 2b

Parameter	Estimate	Standard Error	95% Confidence Limits	
A	0.9601	0.00254	0.9551	0.9650
B	0.000109	0.000017	0.000076	0.000142
C	0.000028	3.959E-6	0.000020	0.000036
D	-0.00002	2.492E-6	-0.00002	-0.00001
θ	9.4812	0.0337	9.4150	9.5474
ϕ_1	-3.3802	0.2525	-3.8754	-2.8851
ϕ_2	-58.5288	0.5126	-59.5339	-57.5236
λ	1.0268	0.0191	0.9893	1.0644

Table 2c

Parameter	Estimate	Standard Error	95% Confidence Limits	
A	0.9625	0.00222	0.9581	0.9669
B	0.000083	0.000014	0.000054	0.000111
C	0.000015	2.563E-6	9.716E-6	0.000020
D	-0.00002	3.019E-6	-0.00002	-0.00001
E	0.000017	2.77E-6	0.000011	0.000022
θ	5.4020	0.0203	5.3621	5.4419
ϕ_1	28.5583	0.1543	28.2558	28.8609
ϕ_2	1.7746	0.2951	1.1960	2.3532
ϕ_3	3371.3	0.4534	3370.4	3372.2
λ	1.0590	0.0222	1.0154	1.1026

Log-periodicity in the intraday *real*-dollar rate.
Results for the one- (Table 2a), two- (Table 2b), and three- (Table 2c) harmonic log-periodic model.

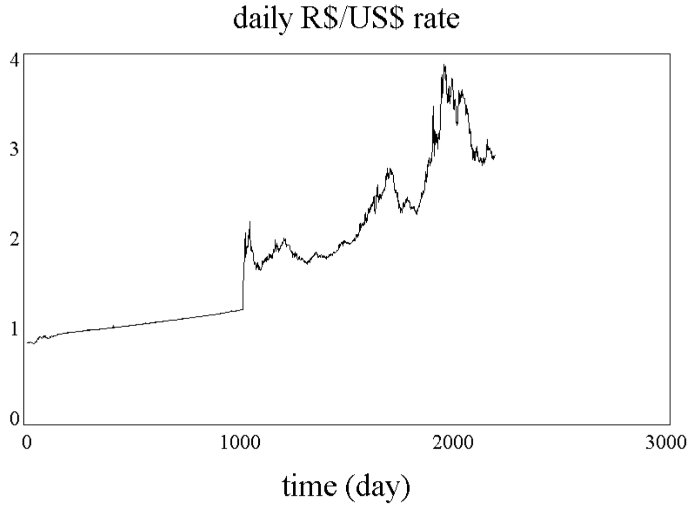


Figure 1a. Daily *real*-dollar rate from 2 January 1995 to 31 Dec 2003.

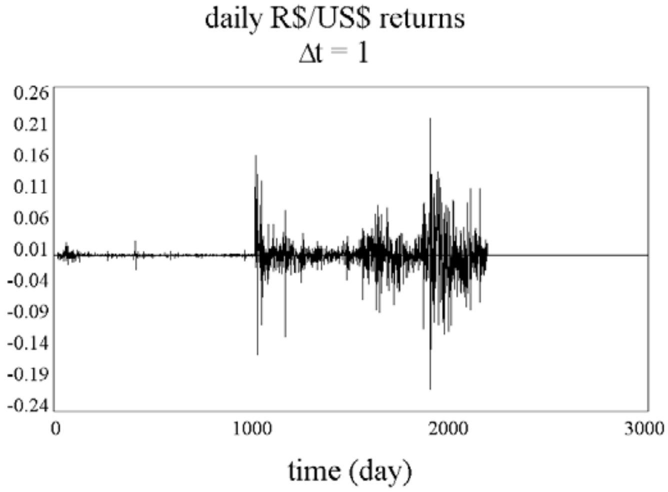


Figure 1b. Daily *real*-dollar single returns ($\Delta t = 1$) from 2 January 1995 to 31 Dec 2003.

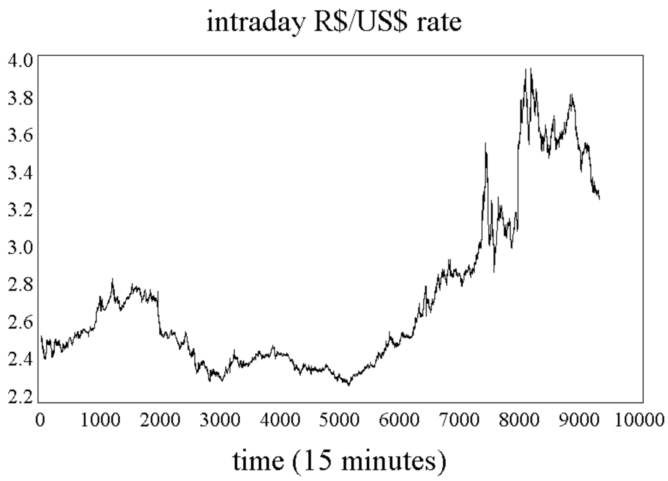


Figure 1c. Fifteen-minute *real*-dollar rate from 9:30 AM of 19 July 2001 to 4:30 PM of 14 January 2003.

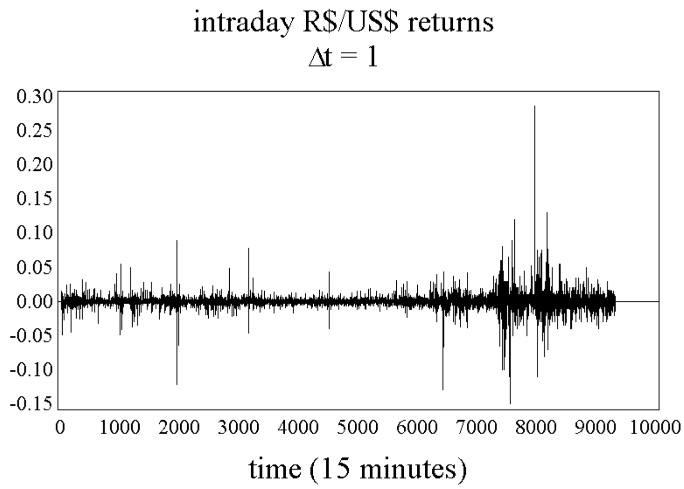


Figure 1d. Fifteen-minute *real*-dollar single returns ($\Delta t = 1$) from 9:30 AM of 19 July 2001 to 4:30 PM of 14 January 2003.

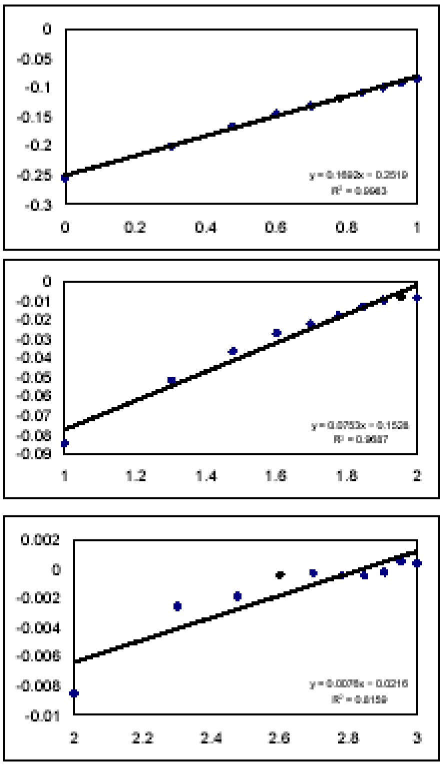


Figure 2a

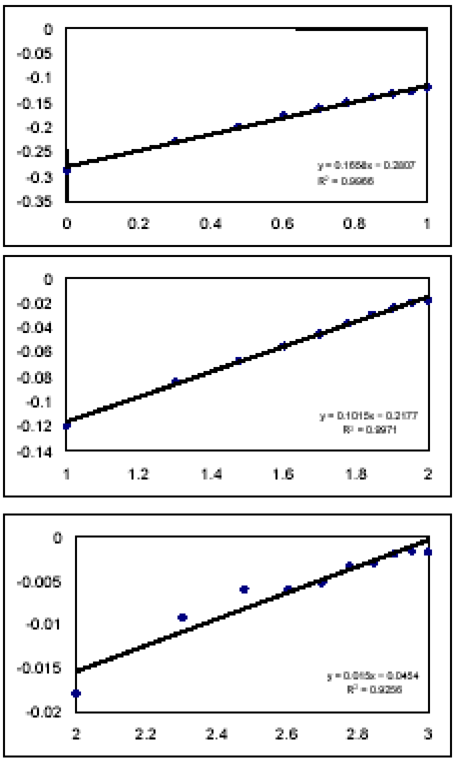


Figure 2b

Power laws in the Hurst exponents for daily (Figure 2a) and intraday (Figure 2b) *real*-dollar returns when time lag is raised in the definition of returns.

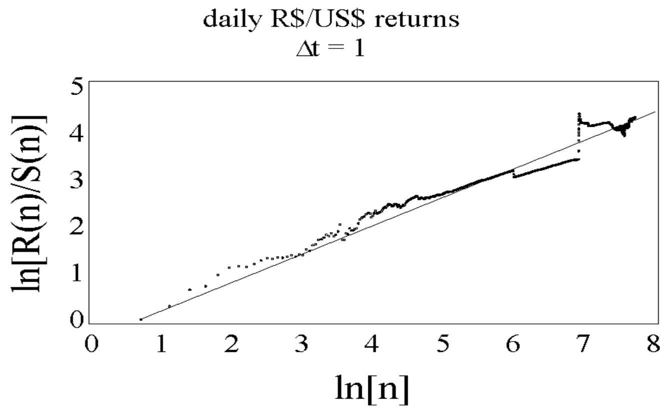


Figure 3a. Hurst exponent for the daily *real*-dollar rate using rescaled range (R/S) analysis. The straight line is the best fit, i.e. $\ln[R(n)/S(n)] = -0.412787 + 0.608028\ln(n)$. A Hurst exponent $H = 0.608028$ is implied.

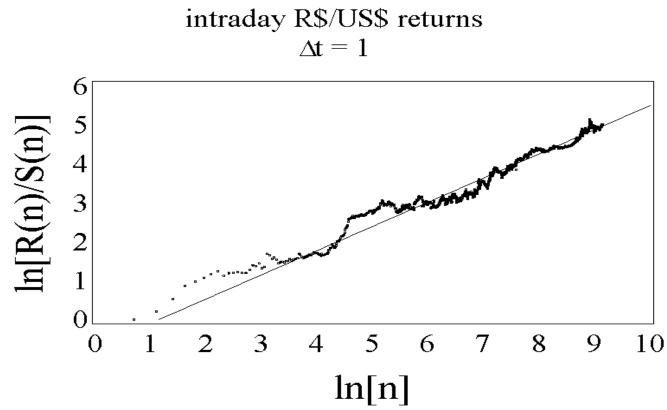


Figure 3b. Hurst exponent for the intraday *real*-dollar rate using rescaled range (R/S) analysis. The straight line is the best fit, i.e. $\ln[R(n)/S(n)] = -0.710114 + 0.622155\ln(n)$. A Hurst exponent $H = 0.62215$ is implied.

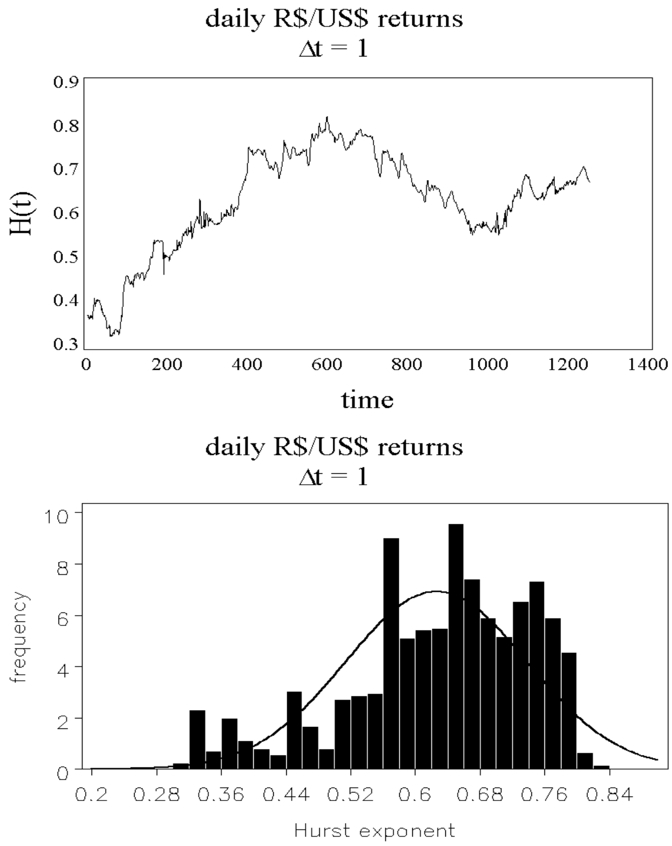


Figure 3c. Time varying Hurst exponents for the daily *real*-dollar rate filtered by an AR(1)-GARCH(1,1) (top), and their histogram (bottom).

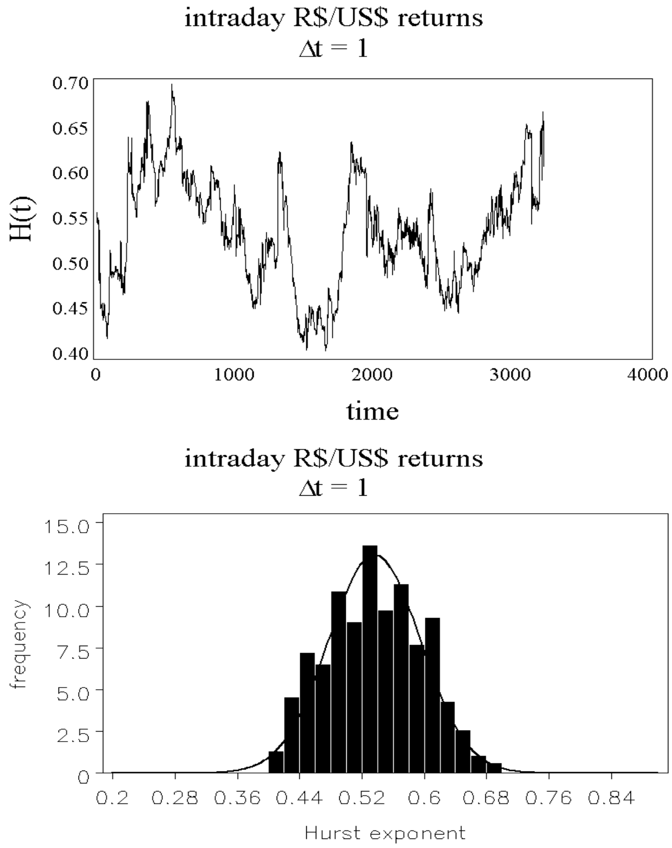


Figure 3d. Time varying Hurst exponents for the intraday *real*-dollar rate filtered by an AR(1)-GARCH(1,1) (top), and their histogram (bottom).

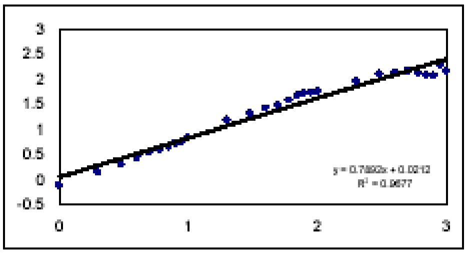


Figure 4a

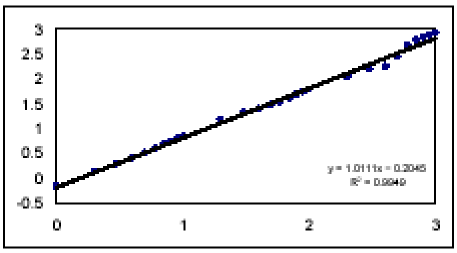


Figure 4b

Power laws in autocorrelation time for increasing lags of the daily (Figure 4a) and intraday (Figure 4b) *real*-dollar returns.

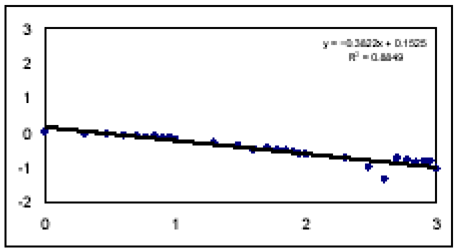


Figure 5a

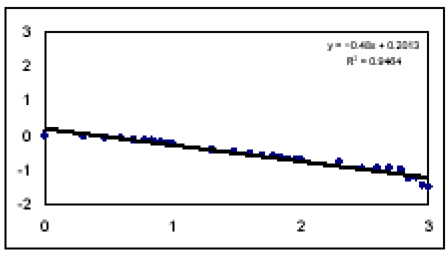


Figure 5b

Power laws in relative *LZ* complexity for increasing lags of the daily (Figure 5a) and intraday (Figure 5b) *real*-dollar returns.

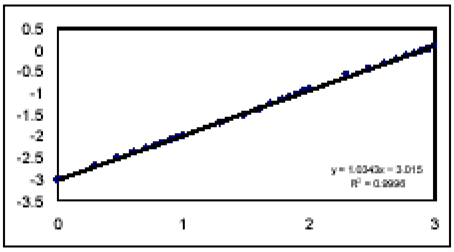


Figure 6a

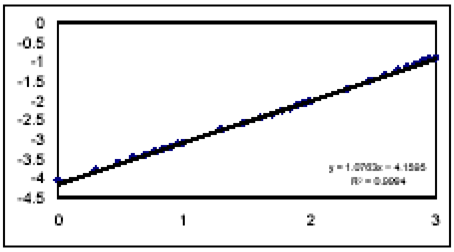


Figure 6b

Power laws in means for increasing lags of the daily (Figure 6a) and intraday (Figure 6b) *real*-dollar returns.

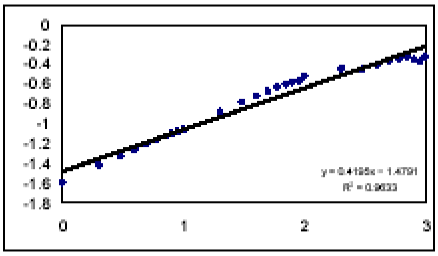


Figure 7a

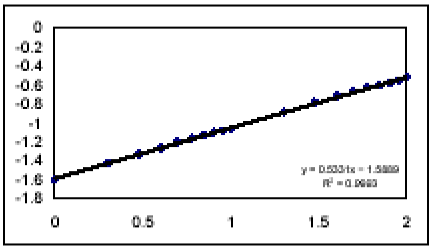


Figure 7b

Power laws in standard deviation for increasing lags of the daily (Figure 7a) intraday (Figure 7b) *real*-dollar returns.

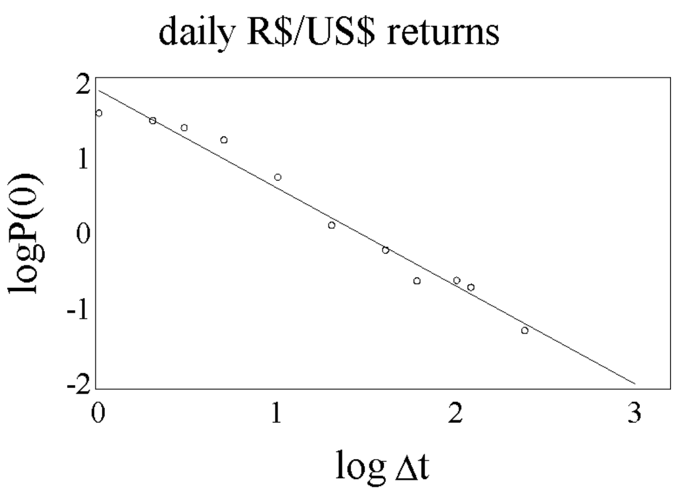


Figure 8a

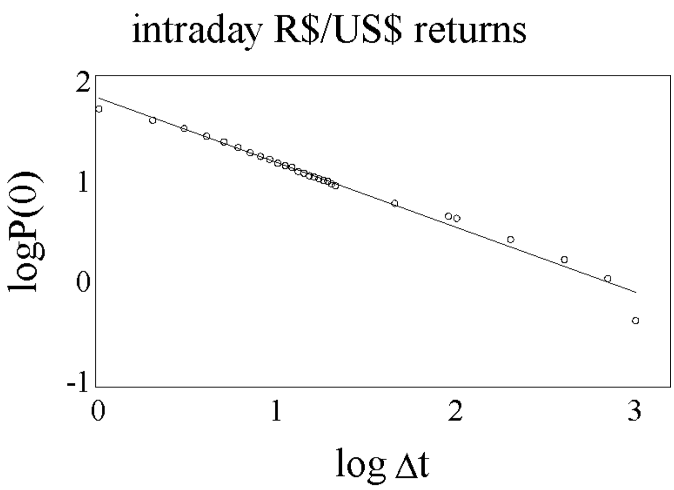


Figure 8b

Log-log plots of the probability of return to the origin $P(0)$ against time lag Δt for the daily (Figure 8a) and intraday (Figure 8b) *real*-dollar returns. Power laws emerge within the time window of $1 \leq \Delta t \leq 1000$. This non-Gaussian scaling is consistent with the presence of a truncated Lévy flight.

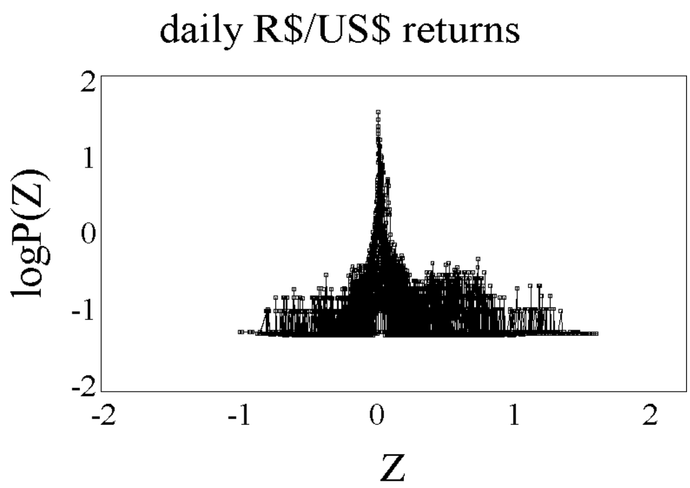


Figure 9a

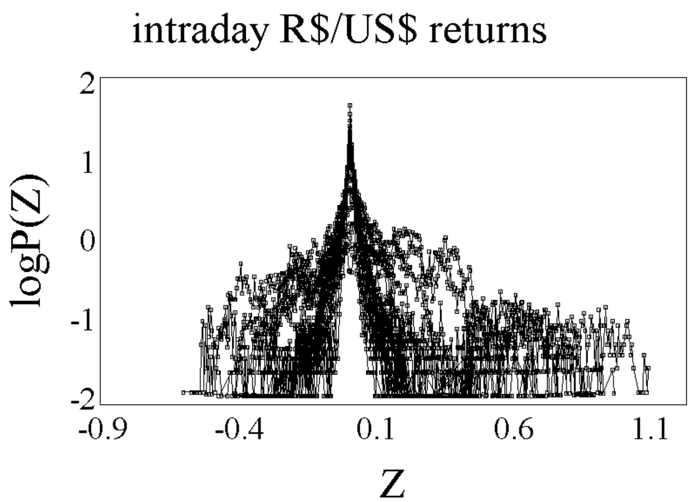


Figure 9b

Logarithm of the PDFs of the daily (Figure 9a) and intraday (Figure 9b) returns of the *real*-dollar rate. A spreading of the PDFs characteristic of any random walk is observed.

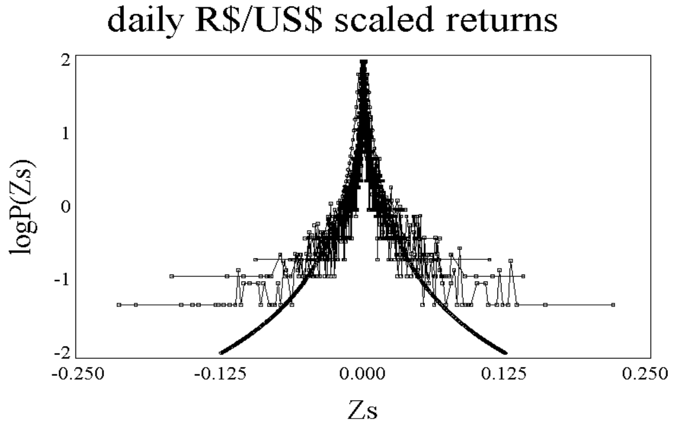


Figure 10a

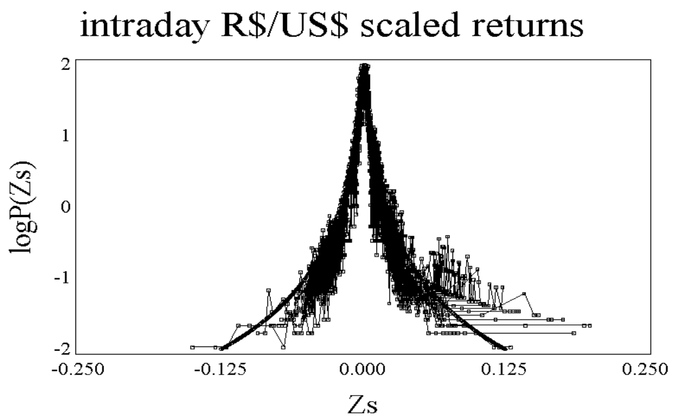


Figure 10b

The same PDFs as in Figures 9a and 9b now plotted in scaled units $P(Z)$. Given the scaling index α, the data are made to collapse onto the $\Delta t = 1$ distribution.

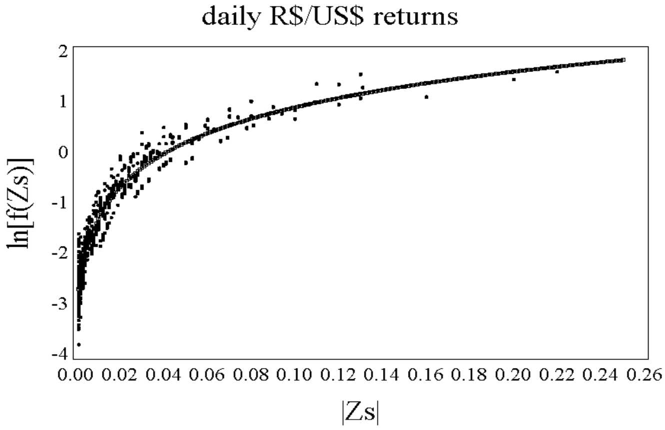


Figure 11a

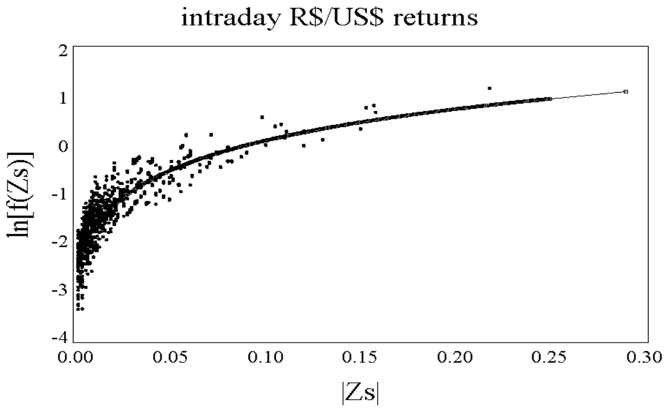


Figure 11b

Log of differences showing how the observed log PDFs of the daily (Figure 11a) and intraday (Figure 11b) *real*-dollar returns deviate from the original log Lévy process. The continuous lines are the fittings using the variance and $Z_s = \Delta t^{-1/\hat{\alpha}} Z_{\Delta t}$.

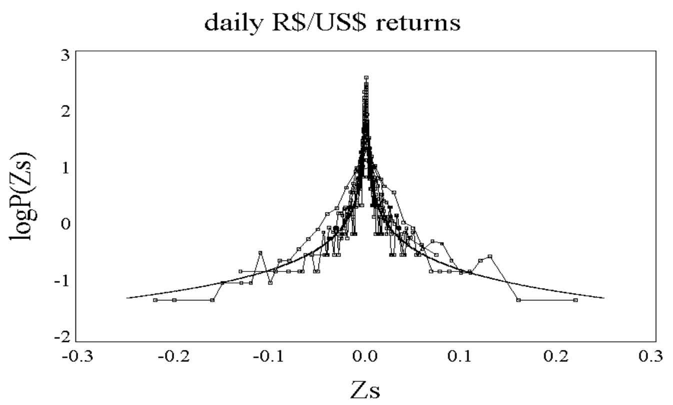


Figure 12a.

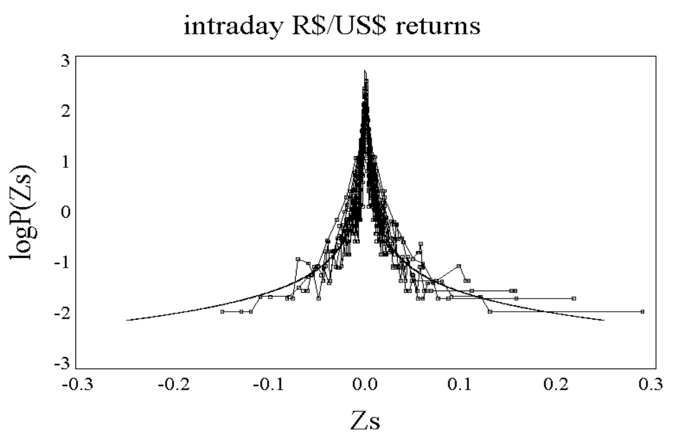


Figure 12b

The same PDFs as in Figures 9a and 9b but now plotted in scaled units $P(Z_s)$, where $Z_s = \Delta t^{-1/\hat{\alpha}} Z_{\Delta t}$. Given the scaling index α for the daily and intraday *real*-dollar returns, the data are made to collapse onto a $\Delta t = 1$ distribution. The curves are the exponentially damped Lévy flights estimated from the data.

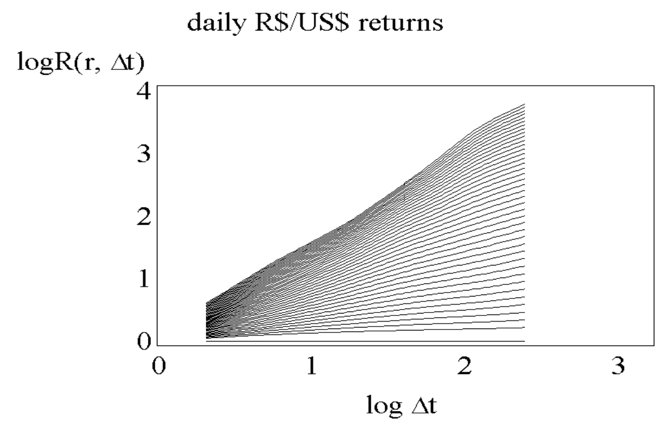


Figure 13a

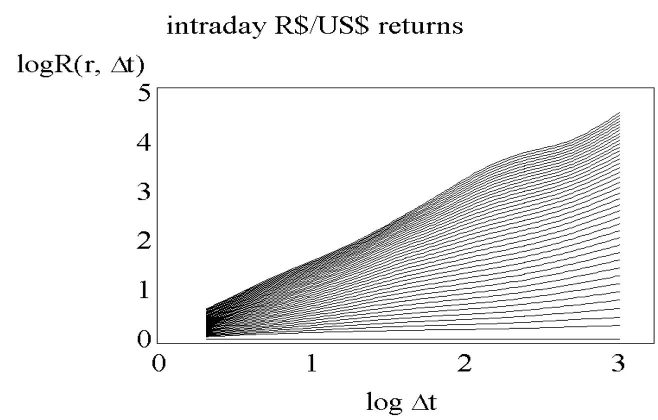


Figure 13b

Estimated ratios $R(r, \Delta t)$ of the daily (Figure 13a) and intraday (Figure 13b) *real*-dollar rate for $r = 0.0$–3.0 at intervals of 0.2. The bottom line corresponds to $r = 0.0$, and the top one to $r = 3.0$.

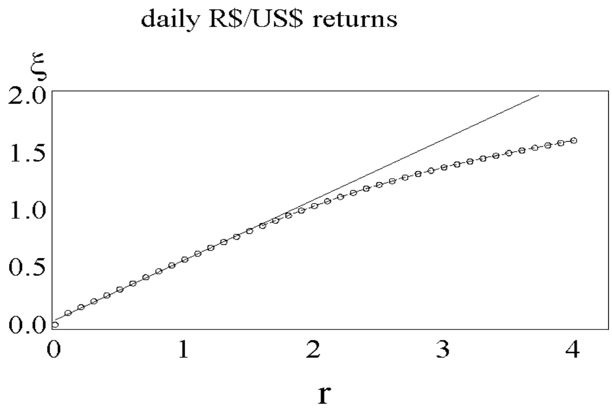


Figure 14a

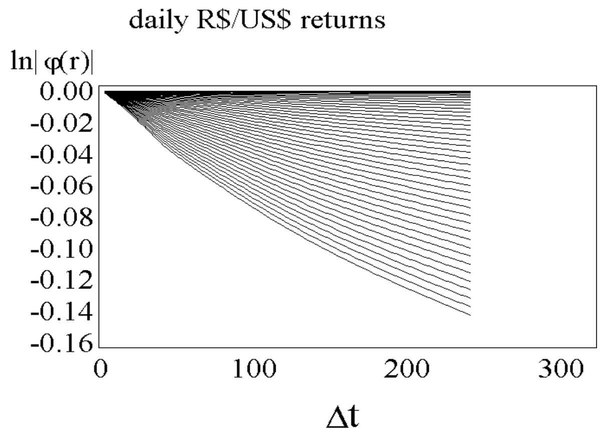


Figure 15a

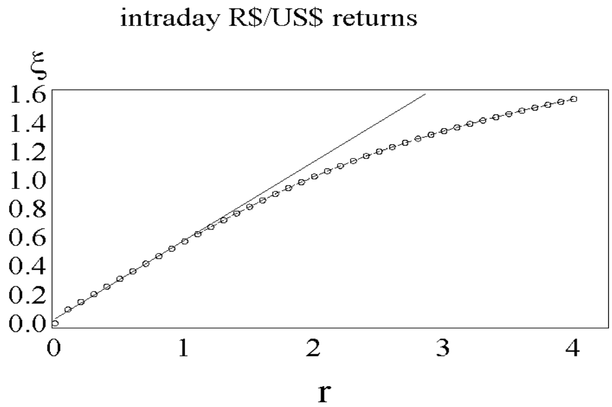


Figure 14b

Estimated multiscaling exponent ξ for the daily (Figure 14a) and intraday (Figure 14b) *real*-dollar rate. An approximate linear behavior for all r gives a piece of evidence of mere single scaling. A linear behavior for initial values of $r < \alpha$ followed by a nonlinear pattern after $r > \alpha$ tracks the presence of multiscaling. As can be seen, the data seem to exhibit multiscaling.

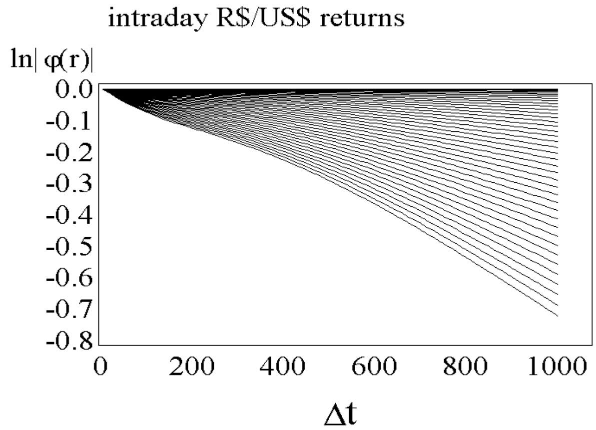


Figure 15b

Estimated ratios $\ln\|\varphi(r)\|$ of the daily (Figure 15a) and intraday (Figure 15b) *real*-dollar returns for r = 0.0–3.0 at intervals of 0.2. For each plot, the upper line corresponds to r = 0.0, and the bottom one to r = 3.0.

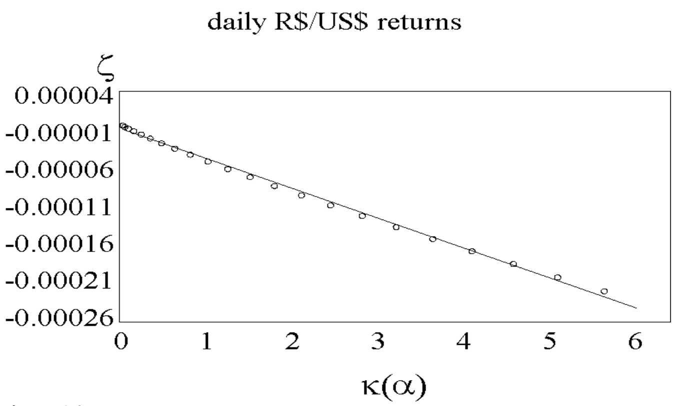


Figure 16a

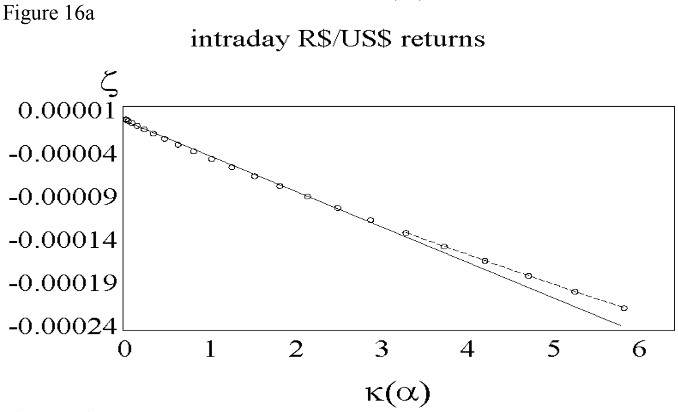
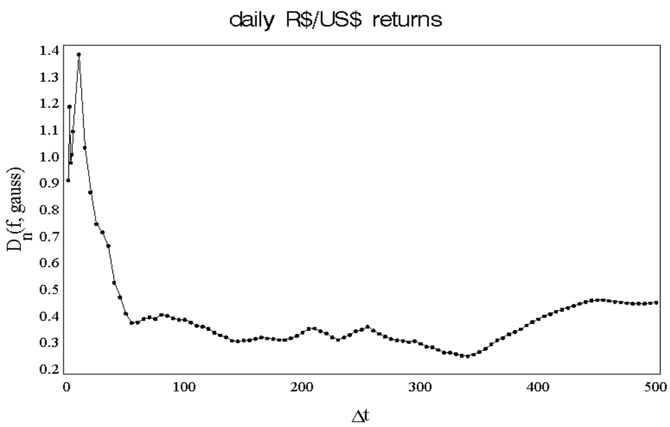


Figure 16b

Estimated multiscaling exponent ζ for the daily (Figure 16a) and intraday (Figure 16b) *real*-dollar returns. An approximate linear behavior for all $\kappa(\alpha) = |r|^{\alpha}$ indicates mere single scaling. A linear behavior for initial values of $\kappa(\alpha) < \alpha_o$ followed by a nonlinear pattern after $\kappa(\alpha) > \alpha_o$ captures the presence of multiscaling. As can be seen, as for ζ multiscaling is arguably absent from the data.



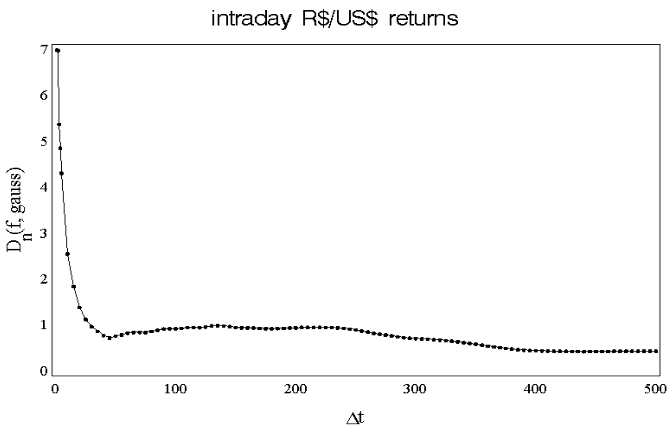


Figure 17. (a) Distance to the Gaussian regime for the daily variation of the *real*-dollar rate and (b) for the intraday data. The intraday data set seems to converge faster to the Gaussian regime.

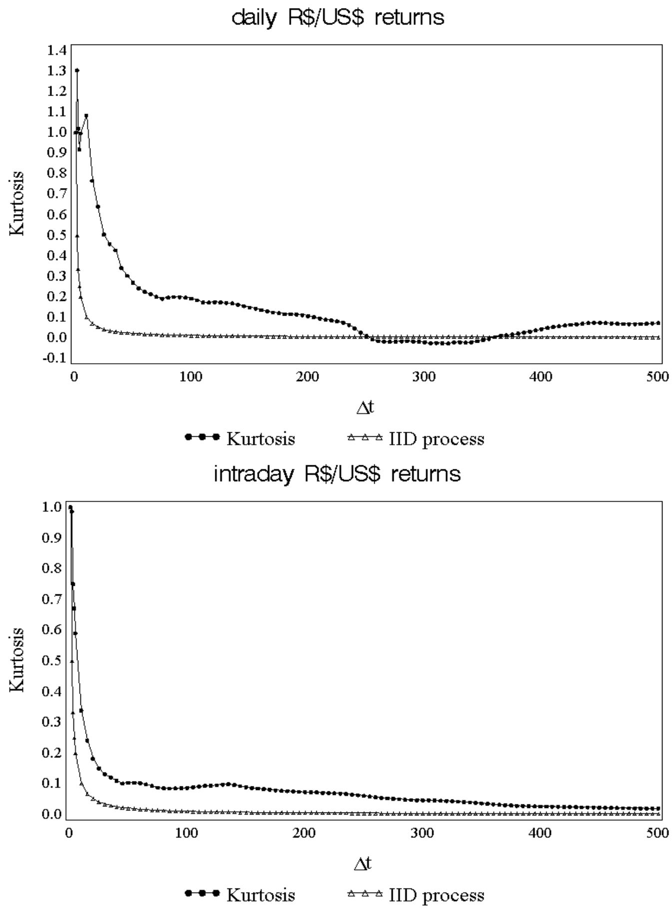


Figure 18. Kurtosis for (a) daily variations and (b) intraday changes of the *real*-dollar rate.

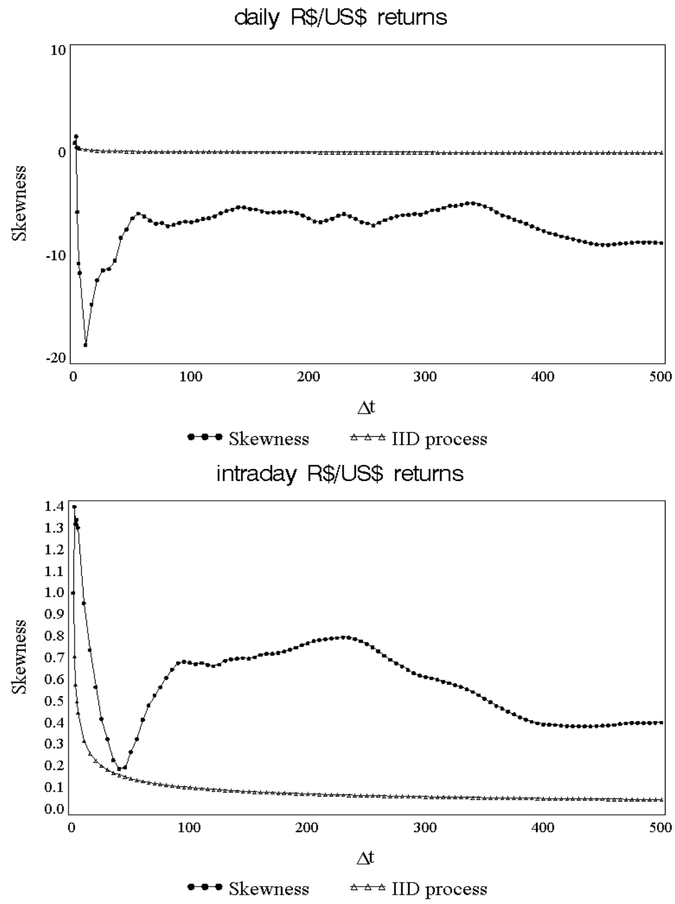


Figure 19. Skewness for (a) daily variations and (b) intraday changes of the *real*-dollar rate.

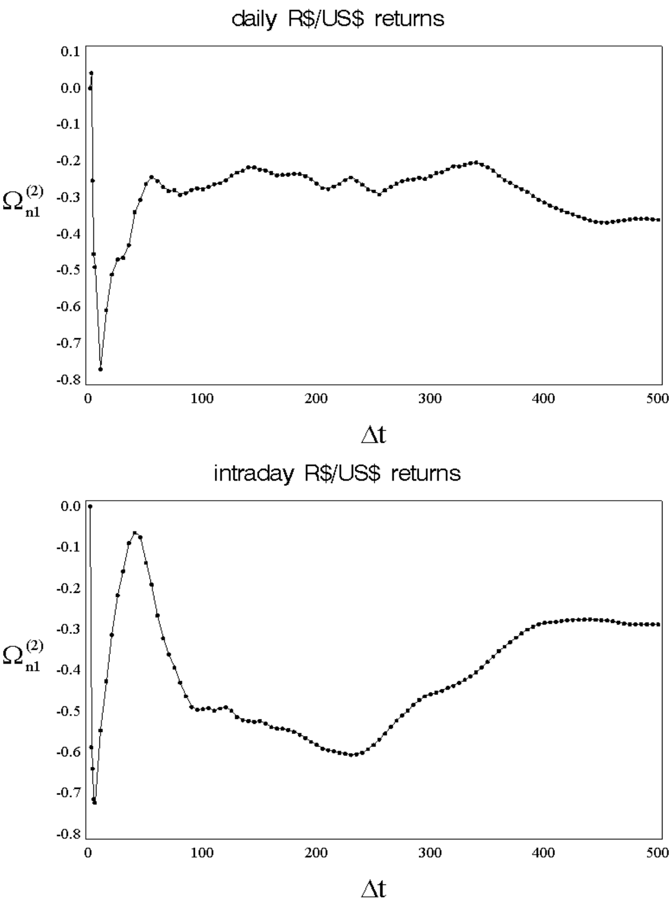


Figure 20. Function $\Omega_{n1}^{(2)}$ for (a) daily variations and (b) intraday changes of the *real*-dollar rate.

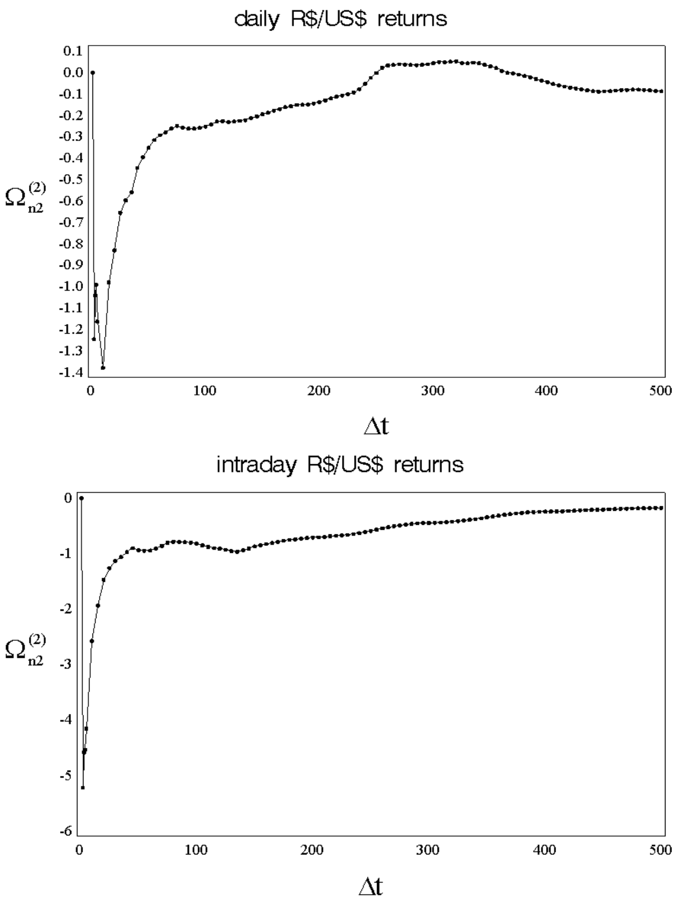


Figure 21. Function $\Omega_{n2}^{(2)}$ for (a) daily variations and (b) intraday changes of the *real*-dollar rate.

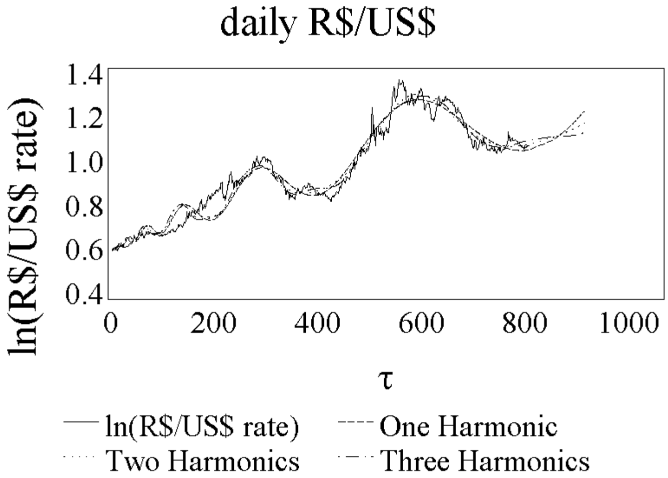


Figure 22a. Log of the daily *real*-dollar rate from 28 August 2000 to 26 September 2003 (continuous line) together with its one-harmonic log-periodic fit (short-dashed line), two-harmonic log-periodic fit (short-dashed line), and three-harmonic log-periodic fit (long-dashed line). The three-harmonic log-periodic formula adjusts better. See Table 1.

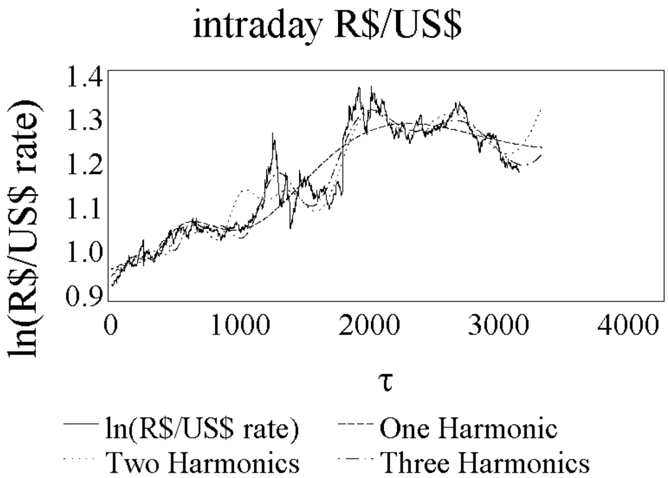


Figure 22b. Log-periodic fits for the intraday *real*-dollar returns. As can be seen, our three-harmonic log-periodic formula adjusts better to the data. Here we have taken 31 May 2002 at 1:00PM as the starting time. See Table 2.