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# The political alliance game: pragmatism, ideology and loyalty

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# The political alliance game: pragmatism, ideology and loyalty

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**Abstract**

Through a sealed-bid first-price auction framework, we model the political parties' choice to ally with each other in order to win an election for an executive office. Leading parties try to entice small ones by simultaneously offering transfers (e.g. government positions, support in other elections, prestige, power of being part of the winning alliance). We highlight the role of three particular reasons for the decision of alliance, namely pragmatism, ideology and alliance loyalty. While the first two may be seen in a simple one-election model, in which two leading parties dispute the support of a small one, loyalty must be analyzed in a dynamic setting with more than one election. Our results for three parties show that, aiming to form an alliance, the party which is favorite in the campaign always tends to offer less transfers to the small one than the underdog does. Furthermore, the closer the leading and the small party are in terms of ideology, the less the transfer offered tends to be. Dynamically, the two effects work jointly to explain the persistence of alliances over time. Finally, we provide two examples which show the complexity of models with more than three parties. The analysis developed here is a first step towards a general model of political alliances.

**Keywords:** elections; political economy; auctions.
**JEL classification:** D72; P48.

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# 1 Introduction

During the Brazilian Presidential campaign of 2002, the domestic political scenario was surprised by the unlikely alliance between the Workers' Party (PT) and Liberal Party (PL)[1]. While PT was one of the largest left-wing Brazilian parties, with a history highly identified with popular movements and known by its anti-capitalism speech, PL was[2] a small right-wing political group founded by businessmen and strongly associated with the defense of taxes cuts, for example. The alliance, which contained other three left-wing parties (PCdoB, PCB and PMN[3]), chose Luiz Inácio Lula da Silva (PT) and José Alencar (PL) as president and vice-president candidates, respectively. Despite the ideological differences between the main two parties, the alliance succeed and won the election receiving 61.27% of the valid cast votes. The great success of the alliance made PT and PL continue the cooperation in the following election in 2006, when the same candidates[4] were reelected.

The example of the presidential elections aforementioned is far from being an exception in Brazilian political scenario. The indifference regarding ideology is even more common in campaigns for offices in regional level (state and municipal), where often parties which are considered to be political enemies by the public ally with each other aiming to win. Consider, for instance, the alliance between the Party of the Liberal Front (PFL, currently Democracts, DEM), a political group considered a symbol of the national right-wingness, and PT in the elections for governor in 2014 in the states of Maranhão, Pará and Paraíba. As an example at a local level, in 2008 the Party of the Brazilian Social Democracy (PSDB), traditional opponent of PT in the presidential elections, has allied with the latter in more than 20% of the Brazilian municipalities. Finally, consider the alliance between PT and the Progressive Party (PP), which, despite the name, is one of the largest right-wing national parties, in the dispute for the mayor of the city of São Paulo in 2012, the largest Brazilian electoral college.

The above examples may suggest that political alliances are always formed pragmatically, with the only objective of winning the election. In fact, the majority of the literature

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[1] It is important to understand the meaning of "liberal" in the Brazilian political context. Parties considered to be liberals in Brazil are, in general, "conservative liberals", such that they are used to combining liberal values and policies with conservative stances, or, more simply, representing the right-wing of the liberal movement.

[2] There are several different classifications of party size. Although when one considers the total of votes received, the number of participation in elections or even the number of candidates elected, PL may be considered a medium party, we choose to classify it as small due to its *relative* size, when compared to the few large parties which dominate the Brazilian political system.

[3] Communist Party of Brazil, Brazilian Communist Party and Party of the National Mobilization, respectively.

[4] In 2006 PL merged with the Party of the National Order Rebuilding (PRONA). The party emerged from that merge was the Party of the Republic (PR), whose ideological orientation did not change, such that it keeps its role in the alliance. In addition, in 2005 José Alencar left PL and joined the Brazilian Republican Party (PRB), a political group that he had helped to found in 2003.

on Brazilian political alliances finds that pragmatism seems to present an increasing force as the main reason of joining a group of parties (Krause and Godoi, 2010; Machado, 2012; Melo, 2015; Miguel and Machado, 2007; Soares, 1964). However, although pragmatism seems to be the strongest force driving the decisions of alliance[5], empirical evidence has reported that in some contexts ideology does have an important role in the choice about whom to ally with (de Lima Júnior, 1983; Lavareda, 1991; Miranda, 2013). Another remarkable empirical evidence is that there is certain persistence in the political alliances over time (Resende and Epitácio, 2017). There are cases in which even when a specific alliance is defeated in the previous election, its members choose to keep it in the future disputes.

In this paper we build a model in which political parties have to decide with whom to ally with in order to win an election for an executive office. Inspired on the Brazilian scenario, we highlight the role of three particular reasons for the decision of alliance, namely pragmatism, ideology and loyalty. We assume that leading parties try to entice small ones by simultaneously offering transfers (e.g. government positions, support in other elections, prestige, power of being part of the winning alliance). Thus, one can see the political alliance game as a sealed-bid first-price auction, in which the leading parties would be the bidders and the small one the object to be disputed. In order to explain the channels through which the three above reasons affect parties' decisions, we explore the auction framework both statically and dynamically.

In a static setting, we provide a baseline model that allows us to analyze (pure) pragmatism. Two leading parties play a game in which each of them makes offers to a small one to convince it to join them in a political alliance. Given that the probabilities of victory of every possible alliance are common knowledge, parties are able to calculate their expected payoffs and thus the small one chooses to ally with the one that offers the highest expected transfers. Our findings show that the party which is favorite in the campaign tends to offer less transfers than the underdog. However, because they can value the office differently, and their offers reflect such preferences, not always the favorite entices the small party. We also show that the closer the ideologies of the underdog party and the small one's, the higher is the transfer offered by the favorite.

When we add dynamics in the baseline model, the determinants of alliance loyalty can be investigated. The idea is that the decisions of alliance made in previous elections can somehow affect the party's current choice about whom to ally with. Our assumption is that a share of the small party voters which have voted for a given alliance in the past becomes loyal to this specific political group. Thus, if in the future such an alliance is dissolved, those voters will not vote for any other party or alliance. Our findings are

[5] The increasing number of political parties in Brazil has been pointed out as the main reason for the decreasing of the importance of the ideology as a determinant of alliance over time (Krause and Godoi, 2010). Consequently, the role of pragmatism has been highlighted even more.

similar in both cases analyzed – namely, with and without predictability about the next election –: if two parties have decided to form an alliance in the past, it is likely that they continue as allied. In fact, the past alliance makes easier – actually, cheaper, in terms of transfers – for the leading party currently to entice the small one.

The complexity of our analysis is largely increased when there is a higher number of parties. While it is still possible to build a general model for the case with many leading parties disputing the support of a single small one, the analysis of settings with more than one small party must be performed on a case-by-case basis. In fact, when there is a single small party and many leading ones, the model may still be seen as a single-object auction, but now with more bidders. However, this is not the case when the number of small parties increases. If one desires to continue to use the auction framework to model the decision of alliance , now it is required to use multiple object ones as well as a "budget constraint". Moreover, a novel strategic relationship among the small parties arises, which makes the analysis richer.

We choose to analyze bilateral rather than block alliances for several reasons. First, leading parties often have power enough to negotiate with small ones individually, after analyzing whether it is worthy trying to entice each one of them. Thus, bilateral alliance may bring a microeconomic intuition about the phenomenon which may be absent when decisions are made by blocks of parties. Second, there is empirical evidence about alliance loyalty only when one considers bilateral partnerships (Krause and Godoi, 2010; Resende and Epitácio, 2017). If we choose to analyze blocks instead of pairs, we would depart from this important stylized fact. Finally, the framework suitable to study block alliances is very similar to the one used in the literature about coalitions, which has already been well developed (Machado, 2009, 2012; Riker, 1962). The analysis of bilateral alliances, on the other hand, is still very incipient, such that our model may be an important contribution to this field.

As we have seen, although there are many examples which would motivate a detailed analysis of the strategic nature of the political alliance decisions, the literature on the subject is virtually null. One can find, for instance, some papers which studies political alliances by adopting a more descriptive approach, specially in the Brazilian context (Melo, 2015; Soares, 1964). Others tries to explore the possible reasons to form alliances, but all those attempts do not provide any formal analysis or game theoretical approach and focus on block partnerships (Krause and Schmitt, 2005). The absence of literature which explores the strategic relationships among parties before elections take place contrasts with the abundance of studies on coalition formation (Budge and Laver, 2016; Machado, 2012; Riker, 1962). However, this latter field is concerned with political decisions after the elections, when the government is already formed and need to form coalitions, something substantially different from the alliances formed before the election with the objective of winning.

There is an important reason for all the above examples and literature to be exclusively related to Brazil, namely the Brazilian political system, which combines two different approaches and it is hardly found elsewhere. The nation's executives and senators are elected through a presidential system: presidents, governors and mayors win by majority run-off, while senators are elected by a plurality of the vote. However, both state and federal deputies are elected according to a system of open list proportional representation: seats are awarded in proportion to the votes that each coalition wins, that is, the candidates who win seats are those who win the most votes within each coalition. Furthermore, the coalition formed in the executive election must hold for the deputies as well. This system, therefore, makes the formation of alliances more beneficial for both leading and small parties. On the one hand, leading parties want to entice the small ones in order to obtain more votes for the executive elections. On the other hand, small parties benefit from being part of an alliance by increasing their chances of having a elected deputy.

Despite the importance of the system of open list proportional representation, it is important to remark that, as we have mentioned, our model does not analyze it. Instead, we focus on the presidential system, in which we can explore the three reason above discussed. In order to take into account the characteristics of a proportional system, a model would have to include simultaneous elections in different levels (e.g. state and federal) for different offices (president, governor, congressman, etc), which is beyond of the scope of this study. Further research on this topic would benefit from incorporating the proportional part of the system.

This paper is organized as follows. The next section presents our baseline model, in which we explore the role of pragmatism in driving the decision of political alliance. In this static version, composed by two leading and one small party, we also analyze the influence of ideology in their alliances. Section 3 adds dynamics by allowing that the three parties run in two consecutive elections. By doing so, we are able to identify the impact of loyalty the parties' behaviors. We extend our model by adding more parties in section 4, and then show the consequent increase in the complexity of the analysis. Section 5 concludes and suggests some extensions.

# 2 Baseline model: political pragmatism

Consider a society where there exist only three political parties, $A$, $B$ and $C$. There will be an election in the next period and they must choose which alliances they will form to win it. None of them individually has the absolute majority of votes, such that an alliance may be essential for assuring the victory. Without loss of generality, we assume that party $A$ is the favorite and party $B$ is the underdog. Party $C$ is a small party which will be disputed by the other two, such that the only two alliances we allow are $A$ and

$C$ (henceforth $AC$), and $B$ and $C$ (henceforth $BC$). We can justify such an assumption by thinking that only the two largest parties have charismatic leaders and thus they are the only able to run in the election as leading of their alliances.

Let $N_i$ be the share (base) of votes for party $i = A, B, C$, then the above assumptions can formally be expressed as $N_A + N_B + N_C = 1$, $1/2 > N_A > N_B > N_C$ and $N_A < N_B + N_C$. This latter inequality means that even though $B$ is the underdog, it wins the election if it succeeds in obtaining all votes of $C$. However, the proportion of voters which follows $C$ in its alliance with other party[6] is a random variable $\mu$. The remaining party $C$ voters – the share $1 - \mu$ – does not cast a ballot in the election (or their votes are invalid). One possible reason why part of the base of $C$ does not follow it and vote for the alliance is ideological: they are loyal only to the party $C$ values and thus do not accept to be part of other political group.

The above assumption creates uncertainty about the election result: if $B$ and $C$ form an alliance, their total share of votes is $N_B + \mu N_C$, which can be lower, higher or equal to $N_A$. Yet it is straightforward to see that there exists $\mu^* \in (0, 1)$ such that $N_B + \mu^* N_C = N_A$ and thus $BC$ wins the election if and only if $\mu > \mu^*$[7]. For future use, observe that $\mu^* = (N_A - N_B)/N_C$. We assume that all the information above is common knowledge. In particular, the base of each party $i$, $N_i$, is public information. The next assumption establishes the distribution of $\mu$, which we also assume to be common knowledge.

**Assumption 2.1** *$\mu$ is a continuous uniform random variable on $[0, 1]$.*

By assuming that $\mu$ is uniformly distributed we can easily obtain the probabilities of victory of each possible alliance:

$$
\begin{aligned}
Pr(\text{AC wins}) &= Pr(N_A + \mu N_C \geq N_B) = Pr\left(\mu \geq \tfrac{N_B - N_A}{N_C}\right) = 1 \\
Pr(\text{B wins}) &= Pr(N_A + \mu N_C < N_B) = Pr\left(\mu \leq \tfrac{N_B - N_A}{N_C}\right) = 0 \\
Pr(\text{BC wins}) &= Pr(N_B + \mu N_C > N_A) = Pr\left(\mu \geq \tfrac{N_A - N_B}{N_C}\right) = 1 - \tfrac{N_A - N_B}{N_C} = 1 - \mu^* \\
Pr(\text{A wins}) &= Pr(N_A \geq N_B + \mu N_C) = Pr\left(\mu \leq \tfrac{N_A - N_B}{N_C}\right) = \mu^*
\end{aligned}
$$

Observe that we do not consider the possibility of $C$ being neutral and not forming alliance with any other party. In fact, as we will see below, the payoff function of $C$ implies that choosing to form an alliance is always preferable to being neutral.

An important remark is that our model does not consider the presence of swing voters. Each voter belongs to a base of some party, that is, he is considered to be a supporter. In addition, voters are not influenced by the platforms announced by candidates. In fact,

[6] In the baseline model we assume that $\mu$ is the same for both parties, that is, the share of party $C$ voters which follows it in an alliance does not depend on the specific ally. In the next section we relax this assumption by allowing ideology to affect the share of voters attracted by each alliance.

[7] Without loss of generality, we assume that in case of tie in the election, $A$ (or its alliance) wins.

the only decision made by parties is whether or not to ally with other political group, such that there is no announcement aiming to influence voters' choice. Our model therefore assumes that the political base of each party is fixed. While such an assumption allows us to focus on the transfers that a leading party may offer to a small one, it precludes an investigation about changes in the platforms as a mechanism to attract potential political allies.

## 2.1 Timing of the game

The political alliance game follows the following steps:

1. The two leading parties try to attract party $C$ to their alliances by making simultaneous offers to it. Those offers are monetary and non-monetary transfers, as we present below.

2. Party $C$ accepts the offer which yields the highest expected payoff.

3. The election takes place and the party (or alliance) with the highest number of votes cast wins.

4. The winning leading party receives its payoffs and, in case of alliance, transfers what it was promised to party $C$.

The game is very similar to a first-price sealed-bid auction, given that the two parties make their "bids" simultaneously and the one whose bid is the highest is the "winner", that is, the one which entices party $C$. In the next section we present the parties' payoffs and analyze the game's equilibrium.

## 2.2 Players

### 2.2.1 Party $C$

The payoff of party $C$ depends only on the transfers it receives from the leading of alliances ($A$ or $B$). Those transfers can include government positions, support in other elections (e.g. at state and local levels), prestige and power of being part of the winning alliance, learning some new electoral technology from the leading party, among others. In fact, it is a very common fact that governments allocate positions such as ministries among political allies as a retribution for their support in the previous election[8]. There is no negotiation among parties, that is, the offer is made by the leading party to $C$ and the

[8]However, this is not the only reason why a government may decide to offer high-level positions to politicians of a different party. This strategy can also be used to form a coalition and increase the strength of the allied base, which in turn increase the relative power of the executive government (Ames, 2002).

only action of the latter is accepting or not. We assume that when the parties announce their offers before the beginning of the campaign, they are able to commit to carrying out their promises if elected[9].

Formally, we assume that party $C$ is risk-neutral and thus its Bernoulli utility function is given by $v(t^i) = t^i$, where $t^i$ is the transfer from party $i$. Notice that by staying neutral its payoff is null, such that it always chooses to form an alliance. Thus party $C$ chooses to ally with $A$ if and only if

$$t^A \geq t^B(1 - \mu^*). \tag{2.1}$$

Given that the election result is uncertain when the alliance $BC$ is formed, the expected payoff of party $C$ in this case must take into account the probability of victory of $BC$. Moreover, if that alliance is defeated, its payoff is zero, because now there are no transfers to be made. An initial conclusion that can be drawn from the behavior of party $C$ is that the higher the advantage of the favorite party the higher must be the transfers the underdog offers to $C$ to attract it to the alliance.

### 2.2.2 Parties $A$ and $B$

Parties $A$ and $B$ are also risk-neutral and maximizes their "net rent", which is the utility of holding office $R^i > 0$ minus the transfer $t^i$ made to party $C$ in case of alliance, for $i = A, B$. As usual, the utility from being in office may be derived from salary and other monetary compensations but also may be due to ego rents, which includes prestige, power and other psychological rewards associated with social status and political influence.

The expected utility of party $A$ is therefore

$$U^A = Pr\left(t^A \geq t^B(1-\mu^*)\right)\left(R^A - t^A\right) + Pr\left(t^A < t^B(1-\mu^*)\right)\mu^* R^A, \tag{2.2}$$

while the expected utility of party $B$ is

$$U^B = Pr\left(t^A < t^B(1-\mu^*)\right)(1-\mu^*)\left(R^B - t^B\right). \tag{2.3}$$

Observe that we use the implicit assumption that whenever the party loses the election its utility is null, since $R^i = 0$ and there is no transfer to be made, that is $t^i = 0$.

Their payoffs take into account all possible results. First, they consider the possibility of alliance $AC$, which occurs with probability $Pr(t^A \geq t^B(1-\mu^*))$. In this case, $AC$ wins – and consequently $B$ loses – with certainty. The payoff of party $A$ and party $B$ are $R^A - t^A$ and zero, respectively. The second possible result is the alliance $BC$, which occurs with probability $Pr(t^A < t^B(1-\mu^*))$. Now, the probability of victory of $A$ is $\mu^*$ while the

[9] It may be hard to envision how such commitments would be possible in a one-shot election, so $C$ might be suspicious of campaign promises. However, leading parties might have an incentive to carry through on their promises if they intend to compete in more than one election and if they care about their reputation for telling the truth.

alliance's is $1 - \mu^*$. In case of victory of $A$, there is no transfer to be made, such that its payoff is $R^A$, while if $BC$ wins, the party $B$ has a net rent of $R^B - t^B$.

## 2.3 Equilibrium

Given the first-price auction framework of our model, we look for a symmetric Nash Bayesian equilibrium. In order to do so, we restrict our attention to the case where the "values" that parties $A$ and $B$ assign to victory in the election, namely $R^i$, are private information. In other words, party $i$ knows its own $R^i$ and only the distribution of the value of the other party $j \neq i$, which it is assumed to be common knowledge information. We detail this assumption below.

**Assumption 2.2** *$R^i$ is a random variable independently and identically distributed on $[0, \bar{R}]$ according to a continuous uniform distribution, for $i = A, B$.*

A well known result in auction theory is that when player's valuations are uniformly distributed, a linear equilibrium exists and is unique. Thus assumption 2.2 allows us to focus on strategies with functional form $t^A = a^A + b^A R^A$ and $t^B = a^B + b^B R^B$, where $a^i, b^i$, for $i = A, B$, are parameters to be determined. It is reasonable to expect that $b^i > 0$, since the larger the utility from being in office the party gets the higher its offer to $C$ must be.

By substituting the linear strategy of party $B$ into the payoff of party $A$, we have the following maximization problem for the latter:

$$\max_{t^A} \frac{1}{\bar{R}}\left[\frac{t^A - a^B(1-\mu^*)}{(1-\mu^*)b^B}\right]\left(R^A - t^A\right) + \left\{1 - \frac{1}{\bar{R}}\left[\frac{t^A - a^B(1-\mu^*)}{(1-\mu^*)b^B}\right]\right\}\mu^* R^A, \tag{2.4}$$

where the probabilities were calculated using assumption 2.2. The first-order condition (FOC) is given by

$$-\frac{1}{\bar{R}}\left[\frac{t^A - a^B(1-\mu^*)}{(1-\mu^*)b^B}\right] + \frac{R^A - t^A}{\bar{R}(1-\mu^*)b^B} - \frac{\mu^* R^A}{\bar{R}(1-\mu^*)b^B} = 0, \tag{2.5}$$

which has as solution

$$t^A = (1-\mu^*)\frac{(R^A + a^B)}{2}. \tag{2.6}$$

One can see in (2.5) the trade-off that party $A$ faces. On the one hand, by increasing $t^A$ its chance of enticing $C$ increases, which makes its victory more likely. On the other hand, higher transfers decrease its disposable rent. Which of those two effects dominates depends primarily on the probability of $A$ winning without alliance. Observe in (2.6) that, regardless the value of $a^B$, the higher the $\mu^*$ the less $t^A$. In fact, its transfer is null when there is certainty of victory ($\mu^* = 1$).

A similar procedure for party $B$ yields

$$\max_{t^B} \frac{1}{\bar{R}} \left[ \frac{t^B (1-\mu^*) - a^A}{b^A} \right] (1-\mu^*) \left( R^B - t^B \right), \tag{2.7}$$

whose FOC is

$$-\frac{1}{\bar{R}} \left[ \frac{t^B (1-\mu^*) - a^A}{b^A} \right] (1-\mu^*) + \frac{(1-\mu^*)^2 \left( R^B - t^B \right)}{\bar{R} b^A} = 0, \tag{2.8}$$

such that the solution is

$$t^B = \frac{R^B}{2} + \frac{a^A}{2(1-\mu^*)}. \tag{2.9}$$

The incentives for party $B$ to increase or decrease the transfers it makes to party $C$ are similar to those of its opponent. Higher transfers make the alliance $BC$ more likely and thus increase its chances of victory but at the same time they decrease its net rent $R^B - t^B$. However, the marginal effect of one unit of transfers on the probability of victory is larger for the party $B$ than for party $A$, given that even with the alliance $BC$ there is no guarantee of success, and without it the defeat is sure. Thus, it is expected that in equilibrium $B$ has more incentives to transfer than $A$.

The parameters' values can be obtained as solution of the system $t^A = a^A + b^A R^A$, $t^B = a^B + b^B R^B$, (2.6) and (2.9). It is straightforward to show that $a^A = a^B = 0$, $b^A = (1-\mu^*)/2$ and $b^B = 1/2$. The game's equilibrium is detailed in the above proposition.

**Proposition 2.3** *The only Bayesian Nash equilibrium of the political alliance game above is*

$$t^A = \frac{(1-\mu^*)}{2} R^A \tag{2.10}$$

$$t^B = \frac{R^B}{2}. \tag{2.11}$$

*In other words, the underdog party offers to party $C$ a transfer which is half of its utility of being in office, while the favorite makes an offer which is less than half of its valuation.*

The above result is somehow expected, since we have already identified that the political alliance game is very similar to a first-price auction. A well-known result of auction theory (Krishna, 2009) is that, under almost the same conditions our model presents (assumption 2.2, in particular), in equilibrium each bidder chooses to bid half of his valuation in a first-price auction. Therefore, our result is different from that only because the favorite party presents a discount factor $1-\mu^* < 1$. Such a discount is higher when party $A$ is more favorite in the election. In particular, when it wins with certainty in any situation, there is no incentive for it to transfer to $C$.

The intuition behind the equilibrium described in proposition 2.3 is simple. Everything else held constant, an alliance with $A$ is always preferable to one with $B$ for party $C$, because of the certainty of victory. The two leading parties anticipate this behavior, such that party $A$ is aware of its advantage. In fact, an analysis of (2.5) and (2.8) shows that $A$ can obtain the same probability of victory as $B$ offering less than half of its valuation. The difference in their offers may also be visualized when one supposes that $R^A = R^B$: in this case, what is offered by the favorite is less (in terms of magnitude) than it is by the underdog. Finally, it is also expected that the magnitude of its advantage – measured by $\mu^*$ – has an impact on its optimal transfer, namely it must be the case that the higher $\mu^*$ the lower $t^A$, which can easily be confirmed in (2.10).

## 2.4 The influence of ideology

Let us now relax the assumption that the share of party $C$ voters which follows it in an alliance does not depend on the specific ally. For, assume that when $C$ chooses to ally with $A$, a share $\mu_A$ of its voters votes for $AC$. Similarly, let $\mu_B$ be the share of $N_C$ which votes for $BC$ in case of alliance. One reason why $\mu_A$ and $\mu_B$ may be different is the ideological closeness between the voters of party $C$ and those of the potential allies. The idea is that the closer the voters of two parties in terms of ideology the larger the share of them which votes for a potential alliance.

We model the potential heterogeneity between the ideologies of the parties – and of their voters – by assuming that the support of $\mu_A$ and $\mu_B$ depends on a parameter $\lambda \in (0,1)$, which measures the relative ideological closeness between parties $B$ and $C$. Thus, the higher the $\lambda$ the closer the ideologies of the voters of party $B$ and party $C$, when compared to the proximity between the ideologies of the voters of parties $A$ and $C$. We detail that assumption below.

**Assumption 2.4** *$\mu_A$ and $\mu_B$ are continuous uniform random variables on $[0, 1-\lambda]$ and $[\lambda, 1]$, respectively, where $\lambda \in (0,1)$. Moreover, $\mu_A$ and $\mu_B$ are independent.*

Observe that the higher the $\lambda$ the more likely the realization of high values (close to one) of the random variable $\mu_B$ and the realization of low values (close to zero) of $\mu_A$. The impact of the ideological closeness on the probabilities of victory can be seen below:

$$Pr\left(N_A + \mu_A N_C \geq N_B\right) = 1 \tag{2.12}$$

$$Pr\left(N_A + \mu_A N_C < N_B\right) = 0 \tag{2.13}$$

$$Pr\left(N_A \geq N_B + \mu_B N_C\right) = Pr\left(\mu_B \leq \mu^*\right) = \tfrac{\mu^*-\lambda}{1-\lambda} \tag{2.14}$$

$$Pr\left(N_A < N_B + \mu_B N_C\right) = 1 - \tfrac{\mu^*-\lambda}{1-\lambda} = \tfrac{1-\mu^*}{1-\lambda}. \tag{2.15}$$

Notice that if $\lambda \geq \mu^*$, then $Pr\left(N_A \geq N_B + \mu_B N_C\right) = 0$ and $Pr\left(N_A < N_B + \mu_B N_C\right) = 1$, such that we would have a very simple strategic interaction. In fact, in this case any leading party able to entice party $C$ assures its victory in the election. The underlying game would be a standard first-price auction and the equilibrium would satisfy $t^i = R^i/2$ for $i = A, B$.

The more interesting case is when $\lambda < \mu^*$. Now, party $C$ chooses to ally with $A$ if and only if $t^A(1-\lambda) \geq t^B(1-\mu^*)$. By using assumption 2.2 we can once again focus on linear strategies. The same procedure used in the baseline case allows us to write the optimization problem of party $A$ as

$$\max_{t^A} \frac{1}{\bar{R}}\left[\frac{t^A(1-\lambda)-a^B(1-\mu^*)}{(1-\mu^*)b^B}\right]\left(R^A-t^A\right)+\left\{1-\frac{1}{\bar{R}}\left[\frac{t^A(1-\lambda)-a^B(1-\mu^*)}{(1-\mu^*)b^B}\right]\right\}\frac{(\mu^*-\lambda)}{1-\lambda}R^A, \tag{2.16}$$

whose solution is given by

$$t^A = \frac{(1-\mu^*)}{2(1-\lambda)}(a^B + R^A). \tag{2.17}$$

Likewise, the party $B$ solves the following program

$$\max_{t^B} \frac{1}{\bar{R}}\left[\frac{t^B(1-\mu^*)-a^A(1-\lambda)}{(1-\lambda)b^A}\right]\frac{(1-\mu^*)}{(1-\lambda)}\left(R^B-t^B\right), \tag{2.18}$$

whose solution is given by

$$t^B = \frac{a^A(1-\lambda)}{2(1-\mu^*)} + \frac{R^B}{2}. \tag{2.19}$$

The optimization problem of both parties is the same as in the baseline case except for the presence of the parameter $\lambda$. One can see that as the ideological closeness between voters of parties $B$ and $C$ increases, the expected payoff of party $A$ decreases while the opposite happens with the payoff of $B$. As will see, this will have impact on the equilibrium transfers. The equilibrium is once again obtained by solving the system with $t^A = a^A + b^A R^A$, $t^B = a^B + b^B R^B$, (2.17) and (2.19). The proposition below analyzes the two possible cases.

**Proposition 2.5** *Consider the political alliance game with ideology. If* $\lambda \in (0, \mu^*)$*, then the only Bayesian Nash equilibrium is*

$$t^A = \frac{(1-\mu^*)}{2(1-\lambda)}R^A \tag{2.20}$$

$$t^B = \frac{R^B}{2}. \tag{2.21}$$

*If* $\lambda \geq \mu^*$*, then the only Bayesian Nash equilibrium is* $t^i = R^i/2$ *for* $i = A, B$*.*

We have already analyzed the trivial case in which $\lambda \geq \mu^*$. Suppose then that $\lambda \in (0, \mu^*)$ and compare the results with those of proposition 2.3. The only difference is the presence of the parameter representing the ideological closeness in the optimal choice of party $A$. In fact, we can observe that the marginal impact of $\lambda$ on $t^A$ is positive while there is no effect on $t^B$. This means that when the ideological views of voters of parties $B$ and $C$ are closer, the ex ante electoral advantage of party $A$ is lower. The channel through which this effect operates is the higher probability of high realized values of $\mu_B$. Finally, given the assumption of $\lambda < \mu^*$, it is always the case that $t^A < R^A/2$. In other words, even with the presence of (moderate, since $\lambda < \mu^*$) ideological closeness between the bases of parties $B$ and $C$, the amount transferred by $A$ to $C$ in case of alliance is always lower than half of its evaluation.

# 3 Repeated elections: the role of alliance loyalty

Suppose now that there will be two consecutive elections and all the remaining assumptions of the baseline model continue to hold. In order to incorporate the idea of alliance loyalty, let us assume that if the alliance which actually was formed in the first election was between parties $C$ and $i$, $i = A, B$, then a share $\delta\hat{\mu}_1 N_C$ of the party $C$ voters became loyal to this alliance, with $\delta \in (0,1)$, and where $\hat{\mu}_1$ is the realization of $\mu$ in the first election[10]. This means that those voters only vote for the alliance $iC$, such that in case of $jC$, with $i \neq j$, they does not cast a ballot in the following election (or their votes are invalid). The practical impact of this new assumption is that if in the first election the alliance formed was the one between $A$ and $C$, there is only $(1 - \delta\hat{\mu}_1)N_C < N_C$ supporters of $C$ who can be enticed by $B$.

The existence of $\delta$ may be due to the influence of ideology on voters over time, for example. In order to understand this possibility, observe that when in the first election the alliance $AC$ is formed, a share $\hat{\mu}_1$ of $C$ supporters votes for $A$ and therefore they end up having more contact with party $A$. Such a contact may make them know the ideology and values of party $A$, something possibly unknown for them until that moment. Part of those voters – namely, $(1 - \delta\hat{\mu}_1)N_C$ – may find the ideology and platform of $A$ somehow close to that of $C$, their original party. This similarity between the ideologies makes those voters loyal to the alliance and thus they will only vote for it in the future elections.

Let $\mu_s$ be the share of $C$ voters which follows their party in an alliance in the period $s = 1, 2$. Then, we make the following standard assumptions.

**Assumption 3.1** *$\mu_1$ and $\mu_2$ are independent and continuous uniform random variables on $[0,1]$.*

[10]Other possibility, which we do not analyze here, is that some of the $i$ voters did not like the alliance $iC$. This may also be due to ideological reasons. By knowing the party $C$ better, they can conclude that its ideology and values are different from those of their original party and thus decide not to vote in the alliance $iC$ in the future. The inclusion of this ingredient does not chance qualitatively the results of our model.

**Assumption 3.2** *$R_s^i$ is a random variable independently and identically distributed on $\left[0, \bar{R}\right]$ according to a continuous uniform distribution, for $i = A, B$ and $s = 1, 2$.*

We consider two cases which we believe represent the main two electoral environments. First, we suppose that parties do not anticipate the future consequences – namely, those in the second election – of their choices about with whom to ally with in the first election. This case represents well some societies where there is almost no predictability about the future electoral scenario. Countries such as Brazil present countless changes in the number of political parties over time, with some being created while others are extinct or even merged (Ames, 2002). In such an uncertain environment, it is reasonable to assume that parties are not able to anticipate the number and which groups will run in the next election.

The second case is when the number and identity of the parties in the second election is already known in the first one. Societies with stable electoral scenarios, where it is neither easy nor fast to create a new party, for example, fit this case. Given the above assumption about the loyalty of a share of $C$ voters to the alliance formed in the first election, the choice made in the first period has known – or at least predictable – future consequences. Parties $A$ and $B$ anticipate those consequences and then choose their optimal transfers to party $C$.

As now we have a dynamic model, we use backward induction. In both cases presented above the result of the election in the second period is similar to the baseline model's, because it is the final stage of the game and thus there is no future consequences to be anticipated. The only difference is the presence of the parameter $\delta$, which measures the loyalty degree of $C$ voters to the alliance formed in the first period. Therefore, we can compute

$$\mu_2^{AC} = \frac{N_A - N_B}{(1 - \delta\hat{\mu}_1)N_C} \tag{3.1}$$

$$\mu_2^{BC} = \frac{N_A - N_B}{N_C} = \mu^*, \tag{3.2}$$

where $\mu_2^{AC}$ and $\mu_2^{BC}$ are the minimum share of disposable $C$ voters party $B$ needs to entice in order to win the second election when the alliance in the first period was $AC$ and $BC$, respectively.

It is straightforward to see that $\mu_2^{AC} > \mu_2^{BC}$, given that in case of alliance between $A$ and $C$ in the first election, the maximum number of supporters of party $C$ which may follow it and vote for a potential new alliance $BC$ is lower than in the baseline model. Thus, given that now the "base" has decreased, for $C$ it is necessary to attract a higher share (a higher "rate") of those voters. In fact, $\delta$ may be high enough such that $\mu_2^{AC} \geq 1$, case in which party $B$ loses with certainty in the second election even if it allies with $C$. Henceforth we suppose that $\delta$ is such that $\mu_2^{AC} < 1$ and thus alliance $BC$ still has a chance of winning the election.

Finally, observe that the probabilities of victory are equal to those of the baseline model, such that it suffices to replace $\mu^*$ by either $\mu_2^{AC}$ or $\mu_2^{BC}$, depending on the case. The equilibrium of the second election also is equal to the baseline case, namely $t_2^A = (1-\mu_2^k)R_2^A/2$ and $t_2^B = R_2^B/2$ for $k = AC, BC$, where the index 2 in $R_2^i$ means the second election. We must now analyze the first period optimal choices.

### 3.1 Absence of predictability[11]

Without any predictability about the electoral scenario in the next election, the decision in the first period is independent of the one made in the subsequent one. Therefore the equilibrium is trivial and is the same of the baseline model, given by proposition 2.3. Our main result in this section compares the equilibrium transfers in both elections.

**Proposition 3.3** *In the case of absence of predictability, the two-period political alliance game has the following equilibria:*

*(i)* $\{t_1^A, t_2^A, t_1^B, t_2^B\} = \left\{\frac{(1-\mu^*)R_1^A}{2}, \frac{(1-\mu^*)R_2^A}{2}, \frac{R_1^B}{2}, \frac{R_2^B}{2}\right\}$*, if in the first election the alliance was* $BC$ *and;*

*(ii)* $\{t_1^A, t_2^A, t_1^B, t_2^B\} = \left\{\frac{(1-\mu^*)R_1^A}{2}, \frac{(1-\mu_2^{AC})R_2^A}{2}, \frac{R_1^B}{2}, \frac{R_2^B}{2}\right\}$*, if in the first election the alliance was* $AC$*.*

*Moreover, when the utility of being in office is constant over time, that is,* $R_1^i = R_2^i$*, for* $i = A, B$*, then the second election transfers are non-increasing in time. In particular, when the alliance in the first period was* $AC$*,* $t_1^A > t_2^A$*.*

First, notice that the equilibrium strategy of party $B$ does not change over time, it is always half of the utility of being in office. This happens because there is no effect of their choice in the first election on the second one regardless of whether the alliance was $AC$ or $BC$. In fact, its choice in the second election is independent of the one made in the first. Yet the equilibrium transfers of party $A$ does potentially change over time, given that when the alliance in the first period is $AC$, some voters of the party $C$ become loyal to the alliance, which has consequences in the future elections. However, such a change is exclusively due to the change in the threshold $\mu^k$, $k = AC, BC$. By not anticipating the future consequences, party $A$ adopts the same strategy, but offers less transfers to $C$ in the second than in the first period because now the probability of victory without alliance is higher.

[11] This scenario is actually a particular case of the one in which futures consequences are considered. In fact, when $\beta = 0$ in the next section, we have the same analysis we develop here with absence of predictability. We choose to present it before because the reasons detailed in the text, mainly the similarity to the Brazilian case, our inspiring scenario.

Suppose again that the utility of being in office is the same for both leading parties. In this case, the above result tells us that the favorite party offers less than the underdog in both elections. More interesting, when the favorite actually entices party $C$ in the first election, it offers even less in the future and obtains the same probability of getting $C$ in its alliance. One interpretation of this fact is that alliances tend to be stable over time: once it has been formed, the mechanism through which some voters become loyal to the it makes more likely that the agreements among parties continues in the next elections. Observe that the mechanism we have suggested works through ideology closeness, which seems to be the case in most of the stable political alliances, as we discussed in the introduction.

## 3.2 Taking into account the future consequences

If the parties anticipate the future electoral scenario and the consequences of their behavior in the second election, then they must maximize an intertemporal payoff function. Let $\beta \in (0,1)$ be the intertemporal discount rate, then $U^i = u_1^i + \beta u_2^i$ is the total utility from both elections, where $u_s^i$ is the utility from election $s = 1, 2$ of party $i = A, B$. The backward induction process requires we have the indirect utility of each party in the second period. By using the already obtained optimal choices of the second period, for party $A$ we have

$$\mathbb{E}\left[u_2^A\right] = Pr\left(t_1^A \geq t_1^B(1-\mu^*)\right)\Big[Pr\left(t_2^A \geq t_2^B(1-\mu_2^{AC})\right)\left(R_2^A - t_2^A\right)$$
$$+Pr\left(t_2^A < t_2^B(1-\mu_2^{AC})\right)\mu_2^{AC}R_2^A\Big] + Pr\left(t_1^A < t_1^B(1-\mu^*)\right)\Big[Pr\left(t_2^A \geq t_2^B(1-\mu_2^{BC})\right)\left(R_2^A - t_2^A\right)$$
$$Pr\left(t_2^A < t_2^B(1-\mu_2^{BC})\right)\mu_2^{BC}R_2^A\Big],$$

which can be written as

$$\begin{aligned}\mathbb{E}\left[u_2^A\right] &= \frac{R_2^A}{\bar{R}}\left\{Pr\left(t_1^A \geq t_1^B(1-\mu^*)\right)\left[\frac{R_2^A(1+\mu_2^{AC})}{2} + (\bar{R} - R_2^A)\mu_2^{AC}\right]\right.\\ &+ \left. Pr\left(t_1^A < t_1^B(1-\mu^*)\right)\left[\frac{R_2^A(1+\mu_2^{BC})}{2} + (\bar{R} - R_2^A)\mu_2^{BC}\right]\right\}. \qquad (3.3)\end{aligned}$$

Moreover, the indirect payoff of party $B$ is

$$\begin{aligned}\mathbb{E}\left[u_2^B\right] &= Pr\left(t_1^A \geq t_1^B(1-\mu^*)\right)Pr\left(t_2^A < t_2^B(1-\mu_2^{AC})\right)(1-\mu_2^{AC})(R_2^B - t_2^B)\\ &+ Pr\left(t_1^A < t_1^B(1-\mu^*)\right)Pr\left(t_2^A < t_2^B(1-\mu_2^{BC})\right)(1-\mu_2^{BC})(R_2^B - t_2^B),\end{aligned}$$

which can be writeen as

$$\mathbb{E}\left[u_2^B\right] = \frac{R_2^B(\bar{R}-R_2^B)}{2\bar{R}}\left[Pr\left(t_1^A \geq t_1^B(1-\mu^*)\right)(1-\mu_2^{AC}) + Pr\left(t_1^A < t_1^B(1-\mu^*)\right)(1-\mu_2^{BC})\right]. \tag{3.4}$$

Expressions (3.3) and (3.4) show that parties $A$ and $B$ take into account all possible scenarios (in terms of alliance) and their probabilities of occurrence. Now, they can anticipate those expressions and thus must consider them when they choose their optimal transfers in the first period. Those results are presented in the next proposition.

**Proposition 3.4** *Suppose that the alliance actually formed in the first election was AC, then in the case with predictability about the future electoral scenario, the parties' optimal transfers in the first period of the political alliance game are:*

$$t_1^A = \frac{(1-\mu^*)R_1^A}{2} + \beta\left(\mu_2^{AC}-\mu_2^{BC}\right)\Lambda^A \tag{3.5}$$

$$t_1^A = \frac{R_1^B}{2} + \beta\left(\mu_2^{AC}-\mu_2^{BC}\right)\Lambda^B, \tag{3.6}$$

*where $\Lambda^A$ and $\Lambda^B$ are two positive constants given by*

$$\Lambda^A = \frac{R_2^B\left(\bar{R}-R_2^B\right)+2R_2^A\left(2\bar{R}-R_2^A\right)}{6\bar{R}}$$

$$\Lambda^B = \frac{2R_2^B\left(\bar{R}-R_2^B\right)+R_2^A\left(2\bar{R}-R_2^A\right)}{6\bar{R}(1-\mu^*)}.$$

The novelty here is that when the utility of being in office is constant over time, that is, $R_1^i = R_2^i$, for $i = A, B$, then the second election transfers are decreasing in time for both leading parties. This result is consequence of two facts. The first is that $\Lambda^A, \Lambda^B > 0$, something that follows from the value that parties give to being in office in the second period. The second fact is the difference $\mu_2^{AC} - \mu_2^{BC} > 0$, which we have already seen as consequence of the potential loyalty of party $C$ supporters to the alliance $iC$. In other words, since both parties would like to be in office in the second period and are aware of the importance of the former alliance for achieving it, they have incentives to increase their offers in the first election. In fact, when there is anticipation of the future consequences, the victory in the first election is worthier than in the second period – and than in the baseline model as well.

The results of our model with predictability are qualitatively similar to the previous one. They provide a theoretical explanation for the alliance loyalty over time and do so by analyzing a channel in which ideological closeness make voters loyal to alliances. In fact, the only substantial difference between propositions 3.3 and 3.4 is that in the later the effect of the voters' loyalty is stronger and affects the behavior of both leading parties. As we have mentioned, such difference is expected since the future consequences are anticipated by both of them.

# 4 More parties more complexity

In this section we present two examples of four-parties political alliance games. Rather than investigating their equilibria in detail, we try to highlight the increasing complexity arising from the larger number of players. While the complexity is not so large when the number of leading parties increases, it is quite larger when there is more small parties to be disputed. Moreover, as the number of potential alliances increases, interesting strategic relationships among small parties arise.

## 4.1 Three leading and one small party

Let us consider the one election model once again. Suppose now that there are four parties, namely $A$, $B$, $C$, and $D$, in the society, whose electoral bases satisfy $N_A + N_B + N_C + N_D = 1$, $1/2 > N_A > N_B > N_C > N_D$, $N_A < N_B + N_D$ and $N_A < N_C + N_D$. Only parties $A$, $B$ and $C$ can be leading of political groups, such that we have the following possible alliances: $AD$, $BD$ and $CD$. Observe that the small party, the one which is object of dispute among the leading ones, is now $D$ rather than $C$. The remaining assumptions are the same of the baseline model. In particular, $\mu$ is the same for all parties, that is, the share of $D$ voters which follows it in an alliance does not depend on the specific ally.

While party $A$ is the favorite and therefore is surely elected in case of alliance with $D$, the other two leading ones need a minimum share of $D$ voters in order to win. Formally, we have

$$\mu_B^* = \frac{N_A - N_B}{N_D} < \frac{N_A - N_C}{N_D} = \mu_C^*, \tag{4.1}$$

where $\mu_B^*$ and $\mu_C^*$ are the already known thresholds that measures the minimum share of $D$ voters necessary for the victory of $B$ and $C$, respectively. Thus, the probabilities of victory can easily be computed:

$$Pr(\text{AD wins}) = Pr(N_A + \mu N_D \geq N_B) = 1$$
$$Pr(\text{B wins}) = Pr(\text{C wins}) = 0$$
$$Pr(\text{A wins, given BD}) = Pr(N_A \geq N_B + \mu N_D) = \mu_B^*$$
$$Pr(\text{A wins, given CD}) = Pr(N_A \geq N_C + \mu N_D) = \mu_C^*$$
$$Pr(\text{BD}) = Pr(N_A < N_B + \mu N_D) = 1 - \mu_B^*$$
$$Pr(\text{BD}) = Pr(N_A < N_C + \mu N_D) = 1 - \mu_C^*.$$

The complexity of a model with three leading parties can be seen in their expected payoffs. They now have to consider several different possibilities, as we can observe below:

$$
\begin{aligned}
U^A = & Pr\left(t^A \geq (1-\mu_B^*)t^B \cap t^A \geq (1-\mu_C^*)t^C\right)\left(R^A - t^A\right) \\
& + Pr\left((1-\mu_B^*)t^B > t^A \geq (1-\mu_C^*)t^C\right)\mu_B^* R^A \\
& + Pr\left((1-\mu_B^*)t^B < t^A \leq (1-\mu_C^*)t^C\right)\mu_C^* R^A
\end{aligned} \tag{4.2}
$$

$$U^B = Pr\left(t^A < (1-\mu_B^*)t^B \cap (1-\mu_C^*)t^C < t^B(1-\mu_B^*)\right)(1-\mu_B^*)\left(R^B - t^B\right) \tag{4.3}$$

$$U^C = Pr\left(t^A < (1-\mu_C^*)t^C \cap (1-\mu_C^*)t^C > t^B(1-\mu_B^*)\right)(1-\mu_C^*)\left(R^C - t^C\right) \tag{4.4}$$

Observe, for example, the three possible scenarios considered in the expected payoff of party $A$. First, $A$ may offer transfers higher than those offered by both $B$ and $C$, which makes party $D$ choose to ally with it. In this case, there is certainty of victory, but $t^A$ must be transferred to $D$. The second possibility is that the alliance is $BD$. Now, $A$ wins with probability $\mu_B^*$, but there is no transfer to be made. Similarly, when the alliance is $CD$, $A$ wins with probability $\mu_C^*$, but $t^A = 0$. Payoffs of parties $B$ and $C$ are simpler because the only way they can win is by offering more than the other two opponents.

The next proposition presents the equilibrium of the game with three leading and one small party[12].

**Proposition 4.1** *The only Bayesian Nash equilibrium of the political alliance game with three leading parties and a small one is*

$$t^A = \frac{2(1-\mu_B^* - \mu_C^*)}{3} R^A \tag{4.5}$$

$$t^B = \frac{2R^B}{3} \tag{4.6}$$

$$t^C = \frac{2R^C}{3}. \tag{4.7}$$

*In other words, the underdogs parties offer to party $D$ a transfer which is two thirds of its utility of being in office, while the favorite makes an offer which is less than two thirds of its valuation.*

Given that the assumptions of the baseline model hold in this model as well, the same procedure of looking for a linear symmetric strategies can be used. In fact, once again, there is a strong similarity between the equilibrium of the political alliance game and the sealed-bid first-price auction. Under the same assumptions we made, when there is $N$ bidders, in a standard auction the equilibrium is each bidder offers $(N-1)/N$ of his valuation. Therefore, our model would present the same result if there was no advantage for party $A$, and each one of the three players would offer $(2/3)R^i$, with $i = A, B, C$. It is

[12]The proof of this proposition is quite similar to the previous one, given that all the assumptions of a standard sealed-bid first-price auction model hold, in particular assumption 2.2. Therefore, we decide to omit it.

straightforward to see that the higher the political advantage of party $A$ – measured by the probabilities of winning without alliance – the lower the transfers to party $D$.

## 4.2 Two leading and two small parties

We continue to analyze the scenario in which there are four parties, but now only $A$ and $B$ can be leading of alliances. Consequently, the possible alliances are $AC$, $AD$, $ACD$, $BC$, $BD$ and $BCD$. As in the previous example, we have $N_A + N_B + N_C + N_D = 1$ and $1/2 > N_A > N_B > N_C > N_D$, yet we also assume that there exists $\tilde{\mu} \in (0,1)$ such that $N_A + \tilde{\mu} N_D < N_B + \tilde{\mu} N_C$. This means that the only chances of victory for party $B$ are to ally with either $C$ or $C$ and $D$. In the first case, even if $A$ allies with $D$, $B$ can attract enough $C$ voters to win the election. Notice that we have assumed that the share of voters of $C$ and $D$ which follows their parties in an alliance is the same. In other words, $\mu$ is the same for any alliance[13].

Based on the above discussion, let us define $\mu_* \in (0,1)$ as the value that makes $N_A + \mu_* N_D = N_B + \mu_* N_C$, that is, $\mu_* = (N_A - N_B)/(N_C - N_D)$, and $\mu_{**} \in (0,1)$ such that $N_A = N_B + \mu_*(N_C + N_D)$, that is $\mu_{**} = (N_A - N_B)/(N_C + N_D)$. Clearly, $\mu_* > \mu_{**}$ because when $B$ entices both $C$ and $D$, the share of their voters which is required to win is lower than the one when the alliance is only with $C$. The probability of victory of each possible alliance is given by:

$$
\begin{aligned}
Pr(\text{AC wins}) &= Pr(\text{ACD wins}) = 1 \\
Pr(\text{B wins}) &= Pr(\text{BD wins}) = 0 \\
Pr(\text{AD wins}) &= Pr(N_A + \mu N_D \geq N_B + \mu N_C) = \mu_* \\
Pr(\text{BC wins}) &= Pr(N_A + \mu N_D < N_B + \mu N_C) = 1 - \mu_* \\
Pr(\text{A wins}) &= Pr(N_A \geq N_B + \mu(N_C + N_D)) = \mu_{**} \\
Pr(\text{BCD wins}) &= Pr(N_A < N_B + \mu(N_C + N_D)) = 1 - \mu_{**}.
\end{aligned}
$$

The novelty of this model is the strategic relationship between the two small parties, since one of them accepting the offer of leading party $i$ makes more likely that such an offer is fulfilled not only for itself but also for the other small party. Given that their choices are simultaneous, we can represent the subgame played by parties $C$ and $D$ through a payoff matrix. Before analyzing that, just recall that small parties always choose to ally with the leading party which offers the highest expected transfer. Denote $t_k^i$ as the transfer from leading party $i = A, B$ to small party $k = C, D$. Finally, for the sake of simplicity, we continue to assume that any tie is broken in favor of party $A$.

[13] This simplifying assumption allows us to deal with a single random variable. Given that we already have a source of complexity in our model, namely the larger number of parties, we choose to keep the same $\mu$ for any alliance.

|  |  | Party $D$ |  |
|---|---|---|---|
|  |  | Ally with party $A$ | Ally with party $B$ |
| **Party** $C$ | Ally with party $A$ | $t_C^A, t_D^A$ | $t_C^A, 0$ |
|  | Ally with party $B$ | $t_C^B(1-\mu_*), t_D^A\mu_*$ | $t_C^B(1-\mu_{**}), t_D^B(1-\mu_{**})$ |

Let us find the best response of parties $C$ and $D$. For, first suppose that party $D$ allies with $A$ ($B$). In this case, party $C$ chooses to ally with $A$ ($B$) as well if and only if $t_C^A \geq t_C^B(1-\mu_*)$ ($t_C^A < t_C^B(1-\mu_{**})$, respectively). As $0 < \mu_{**} < \mu_* < 1$, the intuition is similar to the baseline model's, except by the fact that now party $B$ could offer less transfers to $C$ if the alliance with $D$ was guaranteed. Suppose now that party $C$ allies with $A$. Then, $D$ allies with $A$ as well. This is quite intuitive, because when the alliance $AC$ is guaranteed, there is certainty of its victory, such that it is optimal for $D$ to join it and thus receive a positive transfer[14]. If $C$ chooses to ally with $B$ instead, party $D$ chooses to join it if and only if $t_D^A\mu_* < t_D^B(1-\mu_{**})$.

**Lemma 4.2** *Assume that any tie is broken in favor of the party A. Then the subgame represented in the above payoff matrix has the following Nash Equilibria:*

*(i)* $\{A, A\} = \{\textit{Ally with } A, \textit{Ally with } A\}$ *if either* $t_C^A \geq t_C^B(1-\mu_{**})$ *or* $(1-\mu_*)t_C^B < t_C^A < t_C^B(1-\mu_{**})$ *and* $t_D^A\mu^* \geq t_D^B(1-\mu_{**})$;

*(ii)* $\{B, A\} = \{\textit{Ally with } B, \textit{Ally with } A\}$ *if* $t_C^A \geq t_C^B(1-\mu_*)$ *and* $t_D^A \geq t_D^B(1-\mu_{**})$ *and;*

*(iii)* $\{B, B\} = \{\textit{Ally with } B, \textit{Ally with } B\}$ *if* $t_C^A < t_C^B(1-\mu_*)$ *and* $t_D^A \geq t_D^B(1-\mu_{**})$;

One can notice that three of the four possible results may be equilibrium. The only profile of strategies which never is equilibrium is $\{A, B\} = \{\text{Ally with } A, \text{Ally with } B\}$, because it requires that $D$ chooses to ally with $B$ when $C$ chooses $A$, which we have seen that it is not optimal. Based on the lemma 4.2 we can compute the probabilities of occurrence of each equilibrium:

$$Pr(\{A, A\}) = Pr(t_C^A \geq t_C^B(1-\mu_{**})) + Pr(t_C^A \geq t_C^B(1-\mu_*) \cap t_D^A\mu^* \geq t_D^B(1-\mu_{**})) \quad (4.8)$$

$$Pr(\{A, B\}) = 0 \quad (4.9)$$

$$Pr(\{B, A\}) = Pr(t_C^A < t_C^B(1-\mu_*) \cap t_D^A \geq t_D^B(1-\mu_{**})) \quad (4.10)$$

$$Pr(\{B, B\}) = Pr(t_C^A < t_C^B(1-\mu_*) \cap t_D^A < t_D^B(1-\mu_{**})). \quad (4.11)$$

The two leading parties know the above payoff matrix and the conditions for each possible equilibrium. Therefore, their payoffs must take into account all possibilities and

[14]The analysis performed in the previous sections guarantees $t_k^i > 0$ for all $k$ and $i$.

their probabilities of occurrence, as we can see below:

$$U^A = Pr(\{A,A\})(R^A - t_C^A - t_D^A) + Pr(\{B,A\})\mu_*(R^A - t_D^A) \qquad (4.12)$$
$$+Pr(\{B,B\})\mu_{**}R^A$$
$$U^B = Pr(\{B,A\})(1-\mu_*)(R^B - t_C^B) + Pr(\{B,B\})(1-\mu_{**})(R^B - t_C^B - t_D^B) \qquad (4.13)$$

Observe that now each leading party takes part in two different auctions. One of them is similar to the baseline model's, in which they compete for the support of $C$. However, the presence of a fourth party creates a additional auction, in which the support of $D$ is the object. Thus, the suitable framework for the analysis of models such as those of this section is the one of multiple object auctions. Furthermore, as both transfers from the leading party $i = A, B$ depend on how much it values being in office, their total cannot be higher than this valuation, that is, it must be the case that $t_C^i + t_D^i \leq R^i$. This additional characteristic may be understood as if the party had a "budget constraint", given that it cannot spend more than $R^i$.

The two main features cited in the previous paragraphs make it difficult to find and characterize the equilibrium of the game with two leading and two small parties. Since the transfers offered to party $C$ and to party $D$ are connected, there is no straightforward expressions for the probabilities $Pr(\{i,j\})$, for $i, j = A, B$. In fact, the literature (Krishna, 2009) has largely documented the difficulties of studying multiple object models as well as auctions in which bidders are subject to budget constraints. As our objective in this section is just to highlight the complexity of models with more than three parties, we do not address that combination of difficulties in this paper.

Even within the class of models with two leading and two small parties, the complexity may vary greatly according as the distribution of voters. For instance, when $B$ has chance of victory only if it attracts both parties $C$ and $D$, then the payoff matrix is different from the above one and thus the remaining of the model is also different. In particular, in this case there would be only a threshold $\check{\mu} \in (0,1)$ and the expressions of expected payoffs of parties $A$ and $B$ would be shorter. However, the difficulty regarding the multiple object auction with budget constraint remains, and its solution is not a trivial one.

# 5 Concluding remarks

The analysis developed here is a first step towards a general model of political alliances. The three main reasons which may explain the alliance decision of parties, namely pragmatism, ideology and loyalty, are analyzed through a sealed-bid first-price auction framework. Although such a framework allow us to perform an accurate analysis and provides a rich intuition about the strategic behavior of parties, it also presents some limitations which we believe may be overcome in future research and thus generate new insights about the process of alliance formation in politics. Ruling out the assumption of risk neutrality of parties, considering the case of auction with interdependent values – since it is reasonable to assume that parties do not know completely all the sources of benefit and gains of being in office –, and allowing coalition among small parties in order to increase their bargaing power are examples of those extensions.

One of the extensions we believe is particularly promising is to consider other potential sources of alliance loyalty over time. While we focus on the role of the ideology closeness of the parties' bases, which makes part of the voters loyal to a specific alliance, there are other possibilities to be explored. One of them may arise if we rule out the assumption of commitment by leading parties, such that they could promise transfers to small ones but would not fulfill them. In this context, we can assume that the small party is uncertain about how trustworthy the leading parties are before allying with them. By choosing to ally with a specific party in a election, it can observe if the leading party fulfill its promise and then update its belief about the behavior of its ally. Thus, in the next election its alliance decision will be taken based on that information and if, for example, the leading party promised certain transfer and did not fulfill it, probably the small party will not ally with it again.

The uncertainty about how trustworthy the leading parties are may have a even stronger effect on the alliance loyalty over time if the small parties are risk averse. In this case, there would be a resistance to change its current alliance due to the distrust about other potential allies. The idea is that once allied with some leading party, a small one has more information about it than about any other party, which makes the decision of starting a new alliance risky. Finally, if ideology has an important role in the alliance decision, there may be uncertainty about it, such that the real ideology – in contrast to the announced ideology, which can be chosen only to attract voters – of potential allies may considered unknown. The effects would probably be similar to the previous case and would be affected by the risk aversion of the small parties.

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