MODEL-BASED FEEDBACK CONTROL LAW FOR ASYMPTOTIC TRACKING OF TRANSPORT SYSTEMS

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Abstract— This paper is concerned with the set-point tracking of transport systems. Practical engineering applications are continuous annealing furnaces for the tempering of piece goods or gas-heated glass feeders conditioning the liquid glass flow.

To study this problem we consider that these systems are modeled by a first-order hyperbolic partial differential equation with in domain control. We propose a controller based on a geometrical approach and a state-observer, which employs a single sensor on the left boundary of the system. Using the Lyapunov approach we prove that the proposed observer-based control system guarantees the global exponential stability of both the plant and tracking error. Simulation results are provided to illustrate the performance of the proposed method.

Keywords— Control theory, Distributed parameter system, Partial differential equation, Process control, State observer, Transport system.

Resumo— Este artigo trata do seguimento de referência de sistemas de transporte. Aplicações práticas da engenharia são fornos de recocimento contínuo para a temperatura de peças ou alimentadores de vidro na produção de vidro. Para estudar este problema, consideramos que estes sistemas são modelados por uma equação diferencial parcial de primeira ordem com controle no domínio. Propomos um controlador baseado em uma abordagem geométrica e um observador de estados que utiliza um único sensor na condição de contorno esquerda do sistema. Usando um argumento de Lyapunov, provamos a estabilidade exponencial da planta e do erro de seguimento de referência. Resultados de simulação são apresentados para ilustrar o desempenho do método proposto.

Palavras-chave— Teoria de controle, Sistemas com parâmetros distribuídos, Equação diferencial parcial, Controle de Processos, Observador de estados, Sistemas de transporte.

1 INTRODUCTION

In this paper we are concerned with the set-point tracking of transport systems. These systems are described by conduits or conveyor belts and are used for transporting materials between subsequent process stages. Moreover, these systems are equipped with actuators and sensors, in order to modify the material properties, such as temperature, during the transportation process and consequently, improve the production efficiency. Industrial examples are continuous annealing furnaces for the tempering of piece goods (Bitschnau and Kozeck, 2009) and glass feeders (Malchow and Sawodny, 2012).

To study this problem, we consider that these systems are modeled by a first-order partial differential equation (PDE) with an in domain input, where the distributed state is the material property to be controlled. The actuator is modeled by a time dependent control input weighted by a space dependent smooth function. Several industrial processes are typically described in this manner, for example solar power plants (de Andrade et al., 2015), tubular photobioreactors (de Andrade et al., 2016) or continuous furnaces for piecewise goods (Winkler et al., 2009). In these systems, the control objective is to guarantee that the state variable at the end of the spatial domain tracks the set-point value, the challenges of which lie in the infinite dimensionality of the underlying systems (Krstic and Bekiaris-Liberis, 2013).

Different methodologies to control transport systems are available in the literature. In Malchow and Sawodny (2012), a feedforward control design based on the inversion of the system is proposed to control the temperature of industrial glass feeders, whereas in Winkler et al. (2009) a feedforward controller using the method of characteristics is presented. A more general feedforward control design for transport systems is described in Alt et al. (2016). A passivity-based control of a float-glass process is presented in Ydstie and Jiao (2006). Others alternative approaches to handle the considered control problem can be seen in the references from the cited literature.

In this work, we aim at developing a new control methodology for set-point tracking of transport systems. This controller is derived following an approach conceptually similar to the one employed for the synthesis of inversion-based controllers for ordinary differential equation (ODE) systems. As will be shown, the proposed control law requires measurements of the state variable in the whole domain of the system. This assumption is not realistic for the considered application, where sensors are expensive and difficult to maintain. Boundary measurement is a much more likely scenario. To overcome this problem, we design an observer estimating the distributed state over the whole spatial domain from a single boundary measurement. The exponential convergence of the tracking error and stability of the state estimation error of the observer-based control system is proved by the Lyapunov theory. The developed control method is tested through simulations. It is important to highlight that the method developed in this work may find application in other systems that have the same dynamic
The paper is structured as follows. In Section 2 we detail the problem statement. The control law in case of a fully distributed sensing is derived in Section 3. In Section 4, the state observer is developed. The exponential convergence of the combination of the control law and the state observer is presented in Section 5. We illustrate our result in Section 6 with numerical simulations. The concluding remarks and future works are given in Section 7.

**Notation:** The set of real numbers is denoted by \( \mathbb{R} \). \( \partial_\xi \xi \) (resp. \( \partial_x \xi \)) stands for the partial derivative of the function \( \xi \) with respect to \( t \) (resp. \( x \)). When there is only one independent variable, we denote the derivative by \( \dot{\xi} \). By \( \| \|_{\mathcal{L}_2([0,1], \mathbb{R})} \) we denote the norm in \( \mathcal{L}_2([0,1], \mathbb{R}) \) space, defined by \( \| f \|_{\mathcal{L}_2([0,1], \mathbb{R})} = \int_0^1 |f|^2 dx \) for all functions \( f \in \mathcal{L}_2([0,1], \mathbb{R}) \). Similarly, \( \mathcal{H}^1((0,1], \mathbb{R}) \) is the set of all functions \( f \in \mathcal{H}^1([0,1], \mathbb{R}) \) such that \( \int_0^1 (|f|^2 + |\partial_x f|^2)dx \) is finite. Finally, the class of continuously differentiable functions from \([0,1]\) to \( \mathbb{R} \) is denoted by \( \mathcal{C}^1([0,1]; \mathbb{R}) \).

## 2 CONTROL PROBLEM STATEMENT AND EXISTENCE OF SOLUTIONS

### 2.1 Control problem

The transport system under consideration is illustrated in Figure 1. We assume that the governing model equation of the variable to be controlled is given by the following linear first-order hyperbolic PDE:

\[
\partial_t \xi(x,t) + v \partial_x \xi(x,t) + a \xi(x,t) = \varphi(x) u(t)
\]  

(1)

where \( t \in [0,\infty) \) is the time, \( x \in [0,1] \) is the space\(^1 \), \( \xi \) is the state variable, \( v > 0 \) is the transport velocity and \( a > 0 \) is the loss factor. The system has a control input \( u \), which acts on the system via the smooth function \( \varphi(x) \).

**Assumption 2.1** We assume that function \( \varphi \) is described by a polynomial of order \( n \):

\[
\varphi(x) = \sum_{k=0}^{n} \beta_k x^k
\]

with \( \beta_k \in \mathbb{R} \) for \( k = 0, \ldots, n \).

The initial condition of (1) is

\[
\xi(x,0) = 0
\]

(2)

and the boundary condition is given by

\[
\xi(0,t) = 0
\]

(3)

Given a set-point \( \xi_{\text{out}} \in \mathbb{R} \), the control objective for system (1)-(3) is to provide a feedback law \( u(t) \) such that

\[
\lim_{t \to \infty} (\xi(1,t) - \xi_{\text{out}}) = 0
\]

Without loss of generality and for simplicity, it can always be assumed that by an appropriate change of coordinates, the space domain have length 1.

![Figure 1: Transport system with spatially distributed acting control input and the process stages I and II.](image)

To do that, we define the system output as

\[
y(t) = \int_0^1 \xi(x,t) dx
\]

(4)

and the following set-point profile:

\[
y_{\text{ref}}(t) = \int_0^1 \xi^{\ast}(x,t) dx
\]

(5)

where

\[
\xi^{\ast}(x,t) = \frac{1}{l_0} \int_{\xi_{\text{out}}}^{\xi^{\ast}} \varphi(\theta) e^{\frac{\theta-t}{l_0}} d\theta
\]

(6)

It is straightforward to show that \( \xi^{\ast} \) is a steady-state solution to (1)-(3) for some constant \( u = u^{\ast} \neq 0 \). Indeed, differentiating \( \xi^{\ast} \) with respect to \( x \) yields

\[
\partial_x \xi^{\ast}(x,t) = -\frac{a}{v} \int_0^t \xi_{\text{out}}(t) e^{\frac{\theta-t}{l_0}} \varphi(x) d\theta + \frac{1}{l_0} \xi_{\text{out}} \varphi(x) = -\frac{a}{v} \xi^{\ast}(x) + \frac{u^{\ast}}{v} \varphi(x)
\]

(7)

where the second term in the last equality follows from the fact that \( \frac{d}{dt} = \frac{1}{l_0} \partial_t \xi(x,t)dx \).

We will show in the next section that provided a feedback control law such that

\[
\lim_{t \to \infty} (y(t) - y_{\text{ref}}(t)) = 0
\]

then, consequently \( \xi(1,t) \to \xi_{\text{out}} \).

Moreover, we consider that only the boundary measurement \( \xi(1,t) \) is available to the feedback law. In this sense, we solve the problem of estimating the infinite-dimensional state to solve the output feedback problem corresponding to the above objective.

### 2.2 Existence and well-posedness

The study of solutions of linear hyperbolic PDEs is a classical problem that has been investigated in many references, see for instance Russell (1978). In this section we recall some basic facts about the solution of system (1)-(3).
2.2.1 Continuous solutions

The existence and uniqueness of the solution of (1)-(3) can be proved by using the method of characteristics, which enables us to rewrite the PDE as a set of ODEs. Then, if \( u \) is a two continuously differentiable function of its argument, one can show that the solutions of system (1)-(3) are continuously differentiable with respect to their arguments, i.e., \( \xi(t, x) \in C^1([0, 1] \times [0, \infty), \mathbb{R}) \). Moreover, based on Russell (1978), for any \( t \in [0, \infty) \), any \( t \in C^1([0, 1], \mathbb{R}) \) and any \( u(t) \in L^2([0, t], \mathbb{R}) \cap C^1([0, t], \mathbb{R}) \), there exists a finite constant \( K_0 \) such that

\[
\| \xi(t, x) \|_{L^2([0, 1], \mathbb{R})} \leq K_0 \| u(t) \|_2
\]

where \( u(t) \) denotes the restriction of \( u \) to \([0, t]\).

2.2.2 Generalized Solutions

Referring to the system (1), the linear operator

\[
L \xi = -\nu \frac{\partial \xi}{\partial x} - a \xi
\]

defined on the domain in \( L^2([0, 1], \mathbb{R}) \) consisting of functions \( \xi \in C^1([0, 1], \mathbb{R}) \) which satisfy the boundary condition (3), generates a semigroup of bounded linear operators on \( L^2([0, 1], \mathbb{R}) \). Then, given the initial condition (2), the system (1), (3) has the generalized solution \( \xi \in C^1([0, \infty), \mathbb{R}) \) given by

\[
\xi(t, x) = \int_0^t \mathbb{T}(t - \tau) u(\tau) d\tau.
\]

From the general semigroup theory, it is known that \( \mathbb{T} \) satisfies the following property:

\[
\| \mathbb{T}(t) \|_{L^2([0, 1], \mathbb{R})} \leq Ce^{bt}, \quad t \geq 0
\]

where \( C \geq 1 \) and \( b_0 \) is the largest real part of the eigenvalues of the operator \( \mathbb{T} \). If \( b_0 \) is strictly negative, we will say that the operator (7) generates an exponentially stable semigroup \( \mathbb{T}(t) \). Finally, using the above property it is easy to show that the generalized solution also satisfies (6).

3 CONTROL DESIGN

We seek for a control law represented by the following formal operators:

\[
u = K_p e(t) + K_i \int_0^t e(\tau) d\tau
\]

where \( \mathcal{S} \) is an operator mapping \( L^2 \) into \( \mathbb{R} \), \( s \) is a functional and \( v \) is given by

\[
\nu = K_p e(t) + K_i \int_0^t e(\tau) d\tau
\]

with \( e(t) = y_{ref}(t) - y(t) \) and \( K_p, K_i \in \mathbb{R} \).

Control law (9) is motivated by inversion-based controllers for ODE systems. Its main idea is to assign a desired closed-loop dynamic response of \( y(t) \) and consequently \( \xi(1, t) \to \xi_{out} \) as \( t \to \infty \). This result is shown in the next proposition.

Proposition 3.1 Let \( u(t) \) be defined by

\[
u = K_p e(t) + K_i \int_0^t e(\tau) d\tau
\]

where \( e(t) = y_{ref}(t) - y(t) \) and \( K_p, K_i \in \mathbb{R} \).

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Proof: Part (i). Differentiating (4) with respect to time yields

\[
\dot{y}(t) = -\partial_t \xi + K_p (y_{ref} - y(t)) + \frac{1}{\gamma} \varphi(x)dx
\]

Laplace transformation of this linear ODE leads to

\[
\left( \gamma s^2 + (1 + K_p)s + K_i \right)y(s) = K_i y_{ref}(s).
\]

Stability of (11) follows from Routh’s criterion since all coefficients of the left-hand-side polynomial are positive. Since \( y_{ref} \) is time-invariant, it follows from (11) that \( y(t) \to y_{ref} \) as \( t \to \infty \).

Part (ii). Consider the following Lyapunov functional:

\[
V(t) = \frac{1}{2} \int_0^1 (\xi(x, t) - \xi^*(x))^2 dx.
\]

Its time derivative along trajectories of system
(1)-(3) is given by

\[ V = \int_0^1 (\xi(x, t) - \xi^*(x))(\nu \partial_t \xi(x, t) - a \xi(x, t)) + \varphi(x)(u(t))dx \]

\[ = -\nu \int_0^1 (\xi(x, t) - \xi^*(x))\partial_t(\xi(x, t) - \xi^*)dx - (u^* - u(t)) \int_0^1 \varphi(x)(\xi(x, t) - \xi^*(x))dx - \]

\[ a \int_0^1 (\xi(x, t) - \xi^*(x))^2 dx = -\frac{v}{2} (\xi(1, t) - \xi^*(1))^2 - \Delta(t) - a \int_0^1 (\xi(x, t) - \xi^*(x))^2 dx \]

(12)

where

\[ \Delta(t) = (u^* - u(t)) \int_0^1 (\xi(x, t) - \xi^*(x))dx. \]

Due to the exponential convergence of \( y(t) \), it is straightforward to see that \( \Delta(t) \to 0 \) with exponential convergence as \( t \to \infty \). Moreover, \( e(t) \) and \( \xi(1, t) \) are uniformly continuous, and therefore \( V \) is uniformly continuous as well. We conclude from Barbalat’s lemma that \( \dot{V} \to 0 \) as \( t \to \infty \). It follows from (12) that \( \xi(1, t) \to \xi^*(1) \) as \( t \to \infty \).

Part (iii). Consider the steady-state solution:

\[ \partial_t \xi^*(x) + a \xi^*(x) = \varphi(x)u^*(t) \]

(13)

Introducing the new variable \( \phi(x, t) = \xi(x, t) - \xi^*(x) \), combining (1) and (13) we get the error equation

\[ \partial_t \phi(x, t) + v \partial_x \phi(x, t) + a \phi(x, t) = e(x, t) \]

(14)

\[ \phi(0, t) = 0 \]

(15)

where \( e(x, t) = \varphi(x)(u(t) - u^*) \).

From the results above, we know that \( y(t) \) and \( \dot{y}(t) \) are bounded, it follows that \( \xi(x, t), \partial_t \xi(x, t) \) and \( \phi(x, t) \) are bounded as well, and it is clear that the right-hand side of (14) is bounded and asymptotically vanishing, i.e.,

\[ \sup_{x \in [0, 1]} |e(x, t)| \to 0 \text{ as } t \to \infty. \]

(16)

Note that (14) is a transport equation with constant flow velocity \( v \). By the method of characteristics it can be shown that the effect of initial conditions is exactly zero for \( t > t_1 = \frac{1}{v} \). Due to (16), for any \( \tilde{e} > 0 \) there exists \( t_2 \geq t_1 \) such that \( -\tilde{e} \leq e(x, t) \leq \tilde{e} \) for all \( t \geq t_2 \) and \( x \in [0, 1] \). Consider the system

\[ \partial_t \phi(x, t) + v \partial_x \phi(x, t) + a \phi(x, t) = \tilde{e} \]

\[ \phi(0, t) = 0 \]

defined for \( t \geq t_2 \). It is straightforward to see that with the initial condition \( \phi(x, 0) = f(x), \) where \( f(0) = 0 \) the general solution is \( \phi(x, t) = \tilde{e} \frac{1}{v} (1 - \exp\left(\frac{a}{v} t\right)) \) for all \( x \in [0, 1] \) and \( t \geq t_2 \). A similar result can be derived for \( \phi \) defined by

\[ \partial_t \phi(x, t) + v \partial_x \phi(x, t) + a \phi(x, t) = -\tilde{e} \]

\[ \phi(0, t) = 0 \]

and we have

\[ \frac{-\tilde{e}}{a} (1 - \exp\left(\frac{a}{v} t\right)) \leq \phi(x, t) \leq \frac{-\tilde{e}}{a} (1 - \exp\left(\frac{a}{v} t\right)). \]

Since \( \tilde{e} > 0 \) can be chosen arbitrarily small, we conclude that \( \phi(x, t) \to 0 \) as \( t \to \infty \) for all \( x \in [0, 1] \).

Remark 3.1 The state feedback controller (10) was derived similarly to inversion-based controllers for ODE systems. This is possible because: (i) the system (1)-(3) is well defined as shown in Section 2, and (ii) the input/output belongs to finite dimensional spaces and are distributed in space.

The computation of control law (10) requires measurement of the state \( \xi \) in the whole spatial domain. In practice, distributed measurements of the state are not available, and this need to be estimated. In the next section, we propose and observer design reconstructing the distributed state from a single measurement of \( \xi(1, t) \).

4 OBSERVER DESIGN

We design the observer as a copy of (1), (3) plus an error injection term at the boundary condition:

\[ \partial_t \hat{\xi}(x, t) + v \partial_x \hat{\xi}(x, t) + a \hat{\xi}(x, t) = \varphi(x)u(t) \]

(17)

\[ \hat{\xi}(0, t) = -k(\xi(1, t) - \hat{\xi}(1, t)) \]

(18)

The initial condition for (17)-(18) is given by

\[ \hat{\xi}(x, 0) = \xi_0(x) \]

(19)

where \( \xi_0 \in L^2([0, 1], \mathbb{R}) \). Denoting \( \tilde{\xi}(x, t) = \hat{\xi}(x, t) - \xi(x, t) \), this yields the following observer error dynamics:

\[ \partial_t \tilde{\xi}(x, t) + v \partial_x \tilde{\xi}(x, t) + a \tilde{\xi}(x, t) = 0 \]

(20)
with boundary condition
\[ \dot{\xi}(0, t) = -k \xi(1, t). \] (21)

The design of the observer output injection gain is stated in the following proposition, which is based on the Lyapunov theory.

**Proposition 4.1** Let \( 0 < |k| \leq \exp \left( -\frac{1}{2} \mu \right) \), with \( \mu > 0 \). Then, the observer error dynamics (20)-(21) is exponentially stable.

**Proof:** Consider the following Lyapunov functional:
\[ V(t) = \int_0^1 \exp(-\mu x) \xi^2(x, t) dx. \]

Differentiating \( V \) with respect to time yields
\[ \dot{V}(t) = -2v \int_0^1 \exp(-\mu x) \partial_x \xi^2(x, t) dx - 2a \int_0^1 \exp(-\mu x) \xi^2(x, t) dx \] (22)

Integrating by parts the first term of (22) and substituting the boundary condition (21), we get
\[ \dot{V}(t) \leq - \left( v\mu + 2a \right) \int_0^1 \exp(-\mu x) \xi^2(x, t) dx. \]

This concludes the proof. \( \square \)

5 STABILITY OF THE OBSERVER-BASED CONTROL SYSTEM

We now state the main result of the paper:

**Theorem 5.1** Consider system (1), (3), (17)-(19) and the following control law:
\[ u(t) = \left[ K_p(y_{\text{ref}} - \hat{y}(t)) + K_i \int_0^t (y_{\text{ref}} - \hat{y}(\tau)) d\tau \right] + \int_0^t \left[ (\gamma a - 1) \xi(x, t) + \gamma \partial_x \xi(x, t) \right] dx + \frac{1}{\gamma} \int_0^t \phi(x, \tau) d\tau \] (23)

Then, under the Assumptions 2.1 the system (1), (3), (17)-(19): (i) \( \| \dot{\xi} \|_{L^2([0, 1], \mathbb{R})} \rightarrow 0 \), (ii) \( \xi(1, t) \rightarrow \xi_{\text{out}}(t) \) and (iii) \( \xi(x, t) \rightarrow \xi^*(x) \) for all \( x \in [0, 1] \) as \( t \rightarrow \infty \).

**Proof:** This theorem can be proved by simply noticing that the \( (\xi, \tilde{\xi}) \)-system consists of a cascade of the exponentially stable \( \xi \) system into the \( \tilde{\xi} \) system, which is exponentially stable as well. Finally, using the identity \( \dot{\xi} = \tilde{\xi} - \hat{\xi} \), exponentially stability of the tracking error of (1)-(3) follows. \( \square \)

6 SIMULATION RESULTS

To illustrate our result, we implement the observer-controller scheme on a numerical example with \( v = 0.1 \) and \( a = 0.3 \). The function \( \varphi \) was chosen to be \( \varphi(x) = x^2 + Lx + c \), where \( L = 0.5 \) and \( c = 2 \). The control parameters are \( K_p = 1.05 \), \( K_i = 7 \), \( \gamma = 2 \) and \( k = 0.5 \). These values were found by optimizing the ITSE (integral of the time-weighted square error) index using nonlinear programming, in order to improve the performance of the observer-based control system. The initial condition for the plant was set to zero (in accordance with (2)), while the initial condition for the observer was \( \hat{\xi}(x, 0) = 0.3 \), for all \( x \in [0, 1] \).

The results of the proposed controller for the scenario described above are depicted in Figure 2 and 3. For this numerical simulation, it was chosen \( \xi_{\text{out}}(t) = 1 \) and the set-point profile \( y_{\text{ref}} \) was calculated from (5). As can be seen in the upper graphic of Figure 2, \( y(t) \) asymptotically converges to \( y_{\text{ref}} \) without overshoot.

Figure 3 depicts the evolution of the distributed state \( \xi \) and its estimation \( \hat{\xi} \) with the proposed control law (24). As expected by Theorem 5.1, both \( \xi(t, x) \rightarrow \xi^*(x) \) and \( \hat{\xi}(t, x) \rightarrow \xi^*(x) \) as \( t \rightarrow \infty \). Note that the estimated state \( \hat{\xi} \) (bottom graphic of Figure 3) has a discontinuity along the line \( t = \frac{1}{r} \) due to the error injection term at the boundary condition (see (18)). However, this discontinuity does not affect the closed-loop system, since in control law (24) the state \( \hat{\xi} \) is inside the integral.

7 CONCLUSIONS AND FUTURE WORKS

In this paper we propose a model-based feedback control methodology and a state observer for the set-point
tracking problem of transport systems. The global convergence of both, the state estimation error and set-point tracking, are proved by a Lyapunov argument. It is important to highlight that since the system is linear, the separation principle can be used to design the gains of the controller and the state observer separately. As future works we will extend this methodology for transport systems with multiple actuators and sensors (see for instance Malchow and Sawodny (2012)). Other direction of future work, include the extension of the methodology to handle with input constraints.

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References


